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Martingales in self-similar growth-fragmentations and their connections with random planar maps

Bertoin, Jean ; Budd, Timothy ; Curien, Nicolas ; Kortchemski, Igor

Abstract: The purpose of the present work is twofold. First, we develop the theory of general self-similar growth-fragmentation processes by focusing on martingales which appear naturally in this setting and by recasting classical results for branching random walks in this framework. In particular, we establish many-to-one formulas for growth-fragmentations and define the notion of intrinsic area of a growth-fragmentation. Second, we identify a distinguished family of growth-fragmentations closely related to stable Lévy processes, which are then shown to arise as the scaling limit of the perimeter process in Markovian explorations of certain random planar maps with large degrees (which are, roughly speaking, the dual maps of the stable maps of Le Gall and Miermont in *Ann Probab* 39:1–69, 2011). As a consequence of this result, we are able to identify the law of the intrinsic area of these distinguished growth-fragmentations. This generalizes a geometric connection between large Boltzmann triangulations and a certain growth-fragmentation process, which was established in Bertoin et al. (*Ann Probab*, accepted).

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MARTINGALES IN SELF-SIMILAR GROWTH-FRAGMENTATIONS AND THEIR CONNECTIONS WITH RANDOM PLANAR MAPS

Jean Bertoin [♠] Timothy Budd [♥] Nicolas Curien [♦] Igor Kortchemski [♣]

Abstract

The purpose of the present work is twofold. First, we develop the theory of general self-similar growth-fragmentation processes by focusing on martingales which appear naturally in this setting. As an application, we establish many-to-one formulas for growth-fragmentations and define the notion of intrinsic area of a growth-fragmentation. Second, we identify a distinguished family of growth-fragmentations closely related to stable Lévy processes, which are then shown to arise as the scaling limit of the perimeter process in Markovian explorations of certain random planar maps with large degrees (which are, roughly speaking, the dual maps of the stable maps of Le Gall & Miermont [28]). This generalizes a geometric connection between large Boltzmann triangulations and a certain growth-fragmentation process, which was established in [6].

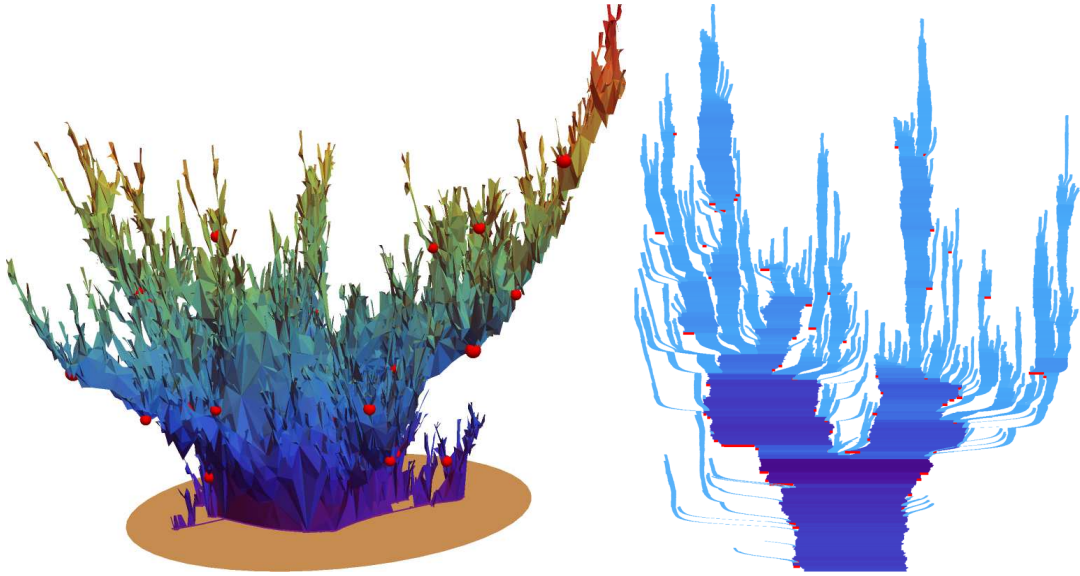


Figure 1: Left: A cactus representation of a random planar map with certain vertices of high degrees (which are the red dots), where the height of a vertex is its distance to the orange boundary. Right: A simulation of the growth-fragmentation process describing the scaling limit of its perimeters at heights (the red part corresponds to positive jumps of the process).

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1 Introduction

In short, the first main purpose of this work is to investigate martingales and supermartingales which arise naturally in the study of self-similar growth-fragmentations. Using them, we identify a remarkable family of growth-fragmentations related to stable Lévy processes, and establish a geometric connection with the stable maps of Le Gall and Miermont [28], which generalizes the one obtained in [6] in the framework of Boltzmann triangulations with a boundary.

Markovian growth-fragmentation processes and cell systems have been introduced in [5] to model branching systems of particles or cells where, roughly speaking, sizes of cells may vary as time passes and then suddenly divide into a mother cell and a daughter cell. More precisely, the sum of the sizes of the mother and of its daughter immediately after a division event always equals the size of the mother cell immediately before division. Further, daughter cells evolve independently one of the others, and follow stochastically the same dynamics as the mother; in particular, they give birth in turn to granddaughters, and so on. Cell systems focus on the genealogical structure of cells, and a growth-fragmentation is then simply obtained as the process of the family of the sizes of cells observed at a given time.

We stress that division events may occur instantaneously, in the sense that on every arbitrarily small time interval, a cell may generate infinitely many daughter cells who can then have arbitrarily small sizes. Division events thus correspond to negative jumps of the mother cell, and even though it was natural in the setting of [5] to assume that the process describing the size of a typical cell had no positive jumps, the applications to random maps that we have in mind incite us to consider here more generally processes which may have jumps of both signs. Only the negative jumps correspond to division events, whereas the possible positive jumps play no role in the genealogy. The latter are only part of the evolution of processes and may be interpreted as a sudden macroscopic growth.

In this work, we only consider self-similar growth-fragmentations, and to simplify we will write *growth-fragmentation* instead of *self-similar growth-fragmentation*. To start with, we shall recall the construction of a cell system from a positive self-similar Markov process (Sec. 2.2). An important feature is that the point process of the logarithm of the sizes of cells at birth at a given generation forms a branching random walk. This yields a pair of genealogical martingales, $(\mathcal{M}^+(n), n \geq 0)$ and $(\mathcal{M}^-(n), n \geq 0)$, which arise naturally as intrinsic martingales associated with that branching random walk (Sec. 2.3). Using classical results of Biggins [13], we observe that \mathcal{M}^+ converges to 0 a.s. whereas \mathcal{M}^- is uniformly integrable; the terminal value of the latter can be interpreted as an intrinsic area.

We then introduce the growth-fragmentation as the process of the sizes of the cells at a given time (Sec. 3.1). We show that its intensity measure can be expressed in terms of the distribution of an associated positive self-similar Markov process via a many-to-one formula (Theorem 3.5 in Sec. 3.2). Using properties of positive self-similar Markov processes, and in particular the fundamental connection with Lévy processes due to Lamperti, we then arrive at a pair of (super-)martingales $M^+(t)$ and $M^-(t)$ indexed by continuous time (Sec. 3.3) and which are naturally related to the two discrete parameter martingales, $\mathcal{M}^+(n)$ and $\mathcal{M}^-(n)$.

Our main purpose in Sect. 4 is to describe explicitly the dynamics of growth-fragmentations under the probability measures which are obtained by tilting the initial one with these intrinsic martingales. Using the well-known spinal decomposition for branching random walks, we show (Theorems 4.2 and 4.7) that the latter can be depicted by a modified cell system, in which, roughly speaking, the evolution of all the cells is governed by the same positive self-similar Markov process, except for the Eve cell that follows a different self-similar Markov process (which, for a negative self-similarity parameter, survives forever in the case of \mathcal{M}^+ and is continuously absorbed in 0 in the case of \mathcal{M}^-).

Once this is done, in Sec. 5 we give different ways to identify the law of a self-similar growth-fragmentation through its cumulant function (Theorem 5.1). Indeed, by [33, Theorem 1.2], the law of a self-similar growth-fragmentation is characterized by a pair (κ, α) , where κ is the so-called cumulant function (see Eq. (5)) and α is the self-similarity parameter. We obtain in particular that the law of a self-similar growth-fragmentation is characterized by the distribution of the process describing the evolution of the Eve cell under the modified probability measure obtained by tilting the initial one with either \mathcal{M}^+ or \mathcal{M}^- . Using this observation, we exhibit a distinguished family of growth-fragmentations which are closely related to θ -stable Lévy processes for $\theta \in (\frac{1}{2}, \frac{3}{2}]$. They are described by a one-parameter family of cumulant functions $(\kappa_\theta; 1/2 < \theta \leq 3/2)$ given by

$$\kappa_\theta(q) = \frac{\cos(\pi(q - \theta))}{\sin(\pi(q - 2\theta))} \cdot \frac{\Gamma(q - \theta)}{\Gamma(q - 2\theta)}, \quad \theta < q < 2\theta + 1.$$

The growth-fragmentation with cumulant function κ_θ and self-similarity parameter $-\theta$ has the property that the evolution of the Eve cell obtained by tilting the dynamics by the martingale \mathcal{M}^+ (resp. \mathcal{M}^-) is the θ -stable Lévy process with positivity parameter ρ satisfying

$$\theta \cdot (1 - \rho) = \frac{1}{2},$$

and conditioned to survive (resp. to die continuously at 0). In the special case $\theta = 3/2$, we recover the growth-fragmentation process without positive jumps that appears in [6].

The final part of the paper (Sec. 6) establishes a connection between the distinguished growth-fragmentations with cumulant function κ_θ (for $1/2 < \theta < 3/2$) and a family of random planar maps that we now describe. Let $\mathbf{q} = (q_k)_{k \geq 1}$ be a non-zero sequence of non-negative numbers. We define a measure \mathbf{w} on the set of all rooted bipartite planar maps by the formula

$$\mathbf{w}(\mathbf{m}) := \prod_{f \in \text{Faces}(\mathbf{m})} q_{\deg(f)/2},$$

where $\text{Faces}(\mathbf{m})$ denotes the set of all faces of a rooted bipartite planar map \mathbf{m} and $\deg(f)$ is the degree of a face f (see Sec. 6.1 for precise definitions). Following [28], we assume that \mathbf{q} is admissible, critical, non-generic and satisfies

$$q_k \underset{k \rightarrow \infty}{\sim} c \cdot \gamma^{k-1} \cdot k^{-\theta-1}$$

for certain $c, \gamma > 0$ and $\theta \in (\frac{1}{2}, \frac{3}{2})$. We then denote by $B^{(\ell)}$ a random bipartite planar map chosen proportionally to \mathbf{w} and conditioned to have a root face of degree 2ℓ (the root face is the face incident to the right of the root edge) and write $B^{(\ell), \dagger}$ for the dual map of $B^{(\ell)}$ (the vertices of $B^{(\ell), \dagger}$ correspond to faces of $B^{(\ell)}$, and two vertices of $B^{(\ell), \dagger}$ are adjacent if the corresponding faces are adjacent in $B^{(\ell)}$). Our assumption implies, roughly speaking, that $B^{(\ell), \dagger}$ has vertices of large degree. The geometry of these maps has recently been analyzed in [16].

We establish that the growth-fragmentations with cumulant function κ_θ appear as the scaling limit of perimeter processes in Markovian explorations (in the sense of [15]) in $B^{(\ell), \dagger}$. In the regime $1 < \theta < 3/2$ (the so-called dilute phase) this connection takes a more geometrical form and we prove that the growth-fragmentation with cumulant function κ_θ and self-similarity parameter $1 - \theta$, which we denote by $\mathbf{X}_\theta^{(1-\theta)}$, describes the scaling limit of the perimeters of cycles obtained by slicing at all heights the map $B^{(\ell), \dagger}$ as $\ell \rightarrow \infty$. This extends [6], where this was shown in the case of random triangulations with the growth-fragmentation $\mathbf{X}_{3/2}^{(-1/2)}$.

In the dilute case $\theta \in (1, \frac{3}{2}]$, we believe that these growth-fragmentations describe the breadth-first search of a one-parameter family of continuum random surfaces which includes the Brownian Map; these random surfaces should be the scaling limit of the dual maps of large random planar maps sampled according to their w -weight and conditioned to be large (see [16] for more details about the geometry of these maps). These questions will be addressed in a future work.

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2 Cell systems and intrinsic martingales

In this section, we start by recalling the construction of a cell system and then dwell on properties which will be useful in this work. As it was mentioned in the Introduction, we shall actually work with a slightly more general setting than in [5], allowing cell processes to have positive jumps. It can be easily checked that the proofs of results from [5] that we shall need here and which were established under the assumption of absence of positive jumps, work just as well when positive jumps are allowed.

The building block for the construction consists in a positive self-similar Markov process $X = (X(t))_{t \geq 0}$, which either is absorbed after a finite time at some cemetery point ∂ added to the positive half-line $(0, \infty)$, or converges to 0 as $t \rightarrow \infty$. We first recall the classical representation due to Lamperti [27], which enables us to view X as the exponential of a Lévy process up to a certain time-substitution. We then construct cell systems based on a self-similar Markov process and derive some consequences which will be useful to our future analysis. In particular, we point at a pair of intrinsic martingales which play a crucial role in our study.

2.1 Self-similar Markov processes and their potential measures

Consider a quadruple $(\sigma^2, b, \Lambda, \mathbf{k})$, where $\sigma^2 \geq 0$, $b \in \mathbb{R}$, $\mathbf{k} \geq 0$ and Λ is a measure on \mathbb{R} such that $\int (1 \wedge y^2) \Lambda(dy) < \infty$ and¹ $\int_{y>1} e^y \Lambda(dy) < \infty$. Plainly, the second integrability requirement is always fulfilled when the support of Λ is bounded from above, and in particular when Λ is carried on \mathbb{R}_- .

The formula

$$\Psi(q) := -\mathbf{k} + \frac{1}{2}\sigma^2 q^2 + bq + \int_{\mathbb{R}} (e^{qy} - 1 + q(1 - e^y)) \Lambda(dy), \quad q \geq 0, \quad (1)$$

is a slight variation of the Lévy-Khintchin formula; it defines a convex function with values in $(-\infty, \infty]$ which we view as the Laplace exponent of a real-valued Lévy process $\xi = (\xi(t), t \geq 0)$, where the latter is killed at rate \mathbf{k} when $\mathbf{k} > 0$. Specifically, we have

$$E(\exp(q\xi(t))) = \exp(t\Psi(q)) \quad \text{for all } t, q \geq 0, \quad (2)$$

with the convention that $\exp(q\xi(t)) = 0$ when ξ has been killed before time t (we may think that $-\infty$ serves as cemetery point for ξ). We furthermore assume that either the killing rate \mathbf{k} of ξ is positive, or that $\Psi'(0+) \in [-\infty, 0)$; that is, equivalently, that 0 is an accumulation point of the open set $\{q > 0 : \Psi(q) < 0\}$. Recall that this is also the necessary and sufficient condition for ξ either to have a finite lifetime, or to drift to $-\infty$ in the sense that $\lim_{t \rightarrow \infty} \xi(t) = -\infty$ a.s. The case $\Lambda((-\infty, 0)) = 0$ when the process X has no negative jumps until it dies, will be uninteresting for our purposes and thus implicitly excluded from now on.

Next, following Lamperti [27], we fix some $\alpha \in \mathbb{R}$ and define

$$\tau_t := \inf \left\{ r \geq 0 : \int_0^r \exp(-\alpha\xi(s)) ds \geq t \right\}, \quad t \geq 0. \quad (3)$$

For every $x > 0$, we write P_x for the distribution of the time-changed process

$$X(t) := x \exp \{ \xi(\tau_{tx^\alpha}) \}, \quad t \geq 0,$$

¹The assumption $\int_{y>1} e^y \Lambda(dy) < \infty$ may be replaced by the weaker assumption that $\int_{y>1} e^{qy} \Lambda(dy) < \infty$ for a certain $q > 0$, but then a cutoff should be added to $q(1 - e^y)$, such as e.g. $q(1 - e^y) \mathbb{1}_{y \leq 1}$ in (1).

with the convention that $X(t) = \partial$ for $t \geq \zeta := x^{-\alpha} \int_0^\infty \exp(-\alpha \xi(s)) ds$. Then $X = (X(t), t \geq 0)$ is both (sub-)Markovian and self-similar, in the sense that for every $x > 0$,

$$\text{the law of } (xX(x^\alpha t), t \geq 0) \text{ under } P_1 \text{ is } P_x. \quad (4)$$

We shall refer here to X as the self-similar Markov process with characteristics $(\sigma^2, b, \Lambda, \mathbf{k}, \alpha)$, or simply (Ψ, α) , as the Laplace exponent Ψ determines the characteristics $(\sigma^2, b, \Lambda, \mathbf{k})$ of the driving Lévy process ξ . We stress that in our setting, either X is absorbed at ∂ after a finite time, or it converges to 0 as time goes to infinity.

We next point at a simple consequence of Lamperti's transformation for the potential measure of X . Recall that for every $x > 0$, the potential measure $U(x, dy)$ is the measure on $(0, \infty)$ defined by

$$\int_{(0, \infty)} f(y) U(x, dy) = E_x \left(\int_0^\zeta f(X(t)) dt \right),$$

where $f : (0, \infty) \rightarrow [0, \infty)$ denotes a generic measurable function.

Lemma 2.1. *Under the assumption above, the Mellin transform of $U(x, dy)$ is determined by*

$$\int_{(0, \infty)} y^{q+\alpha} U(x, dy) = \begin{cases} -x^q / \Psi(q) & \text{if } \Psi(q) < 0, \\ \infty & \text{otherwise,} \end{cases}$$

where (Ψ, α) denotes the characteristics of X .

Proof. It is immediately seen from Lamperti's transformation that

$$\begin{aligned} \int_{(0, \infty)} y^{q+\alpha} U(x, dy) &= E_x \left(\int_0^\zeta X(t)^{q+\alpha} dt \right) = E \left(\int_0^\infty x^{q+\alpha} \exp((q+\alpha)\xi(\tau_{tx^\alpha})) dt \right) \\ &= x^q E \left(\int_0^\infty \exp((q+\alpha)\xi(\tau_s)) ds \right) \\ &= x^q E \left(\int_0^\infty \exp(q\xi(t)) dt \right). \end{aligned}$$

Our assertion then follows from (2) and Tonelli's Theorem. \square

The *negative* jumps of X will have an important role in this work, and it will be convenient to adopt throughout this text the notation

$$\Delta_- Y(t) := \begin{cases} Y(t) - Y(t-) & \text{if } Y(t) < Y(t-), \\ 0 & \text{otherwise} \end{cases}$$

for every càdlàg real-valued process Y . We next set for $q \geq 0$

$$\begin{aligned} \kappa(q) &:= \Psi(q) + \int_{(-\infty, 0)} (1 - e^y)^q \Lambda(dy) \\ &= -\mathbf{k} + \frac{1}{2} \sigma^2 q^2 + bq + \int_{\mathbb{R}} (e^{qy} - 1 + q(1 - e^y) + \mathbb{1}_{\{y < 0\}}(1 - e^y)^q) \Lambda(dy). \end{aligned} \quad (5)$$

Plainly, $\kappa : \mathbb{R}_+ \rightarrow (-\infty, \infty]$ is a convex function; observe also $\kappa(q) < \infty$ if and only if both $\Psi(q) < \infty$ and $\int_{(-\infty, 0)} (1 - e^y)^q \Lambda(dy) < \infty$, and further that the condition $\int_{(-\infty, 0)} (1 \wedge y^2) \Lambda(dy) < \infty$ entails that

$\int_{(-\infty,0)}(1-e^y)^q\Lambda(dy)<\infty$ for every $q\geq 2$. We call κ the *cumulant function*; it plays a major role in the study of self-similar growth-fragmentations through the following calculation done in [5, Lemma 4]:

$$E_x\left(\sum_{0<s<\zeta}|\Delta_-X(s)|^q\right)=\begin{cases}x^q\left(1-\frac{\kappa(q)}{\Psi(q)}\right) & \text{if } \Psi(q)<0 \text{ and } \kappa(q)<\infty \\ \infty & \text{otherwise.}\end{cases}\quad (6)$$

Actually, only the case when X has no positive jumps is considered in [5], however the arguments there works just as well when X has also positive jumps.

By convexity, the function κ has at most two roots. We shall assume throughout this work that there exists $\omega_+>0$ such that $\kappa(q)<\infty$ for all q in some neighborhood of ω_+ and

$$\kappa(\omega_+)=0 \quad \text{and} \quad \kappa'(\omega_+)>0. \quad (7)$$

This forces

$$\inf_{q\geq 0}\kappa(q)<0, \quad (8)$$

and since $\Psi\leq\kappa$, (8) is a stronger requirement than $\Psi(0)=-k<0$ or $\Psi'(0+)<0$ that we previously made. Further, note that in the case when $\Psi(q)<\infty$ for all $q>0$ (which holds for instance whenever the support of the Lévy measure Λ is bounded from above) and the Lévy process ξ is not the negative of a subordinator, then the Laplace exponent Ψ is ultimately increasing, so $\lim_{q\rightarrow\infty}\kappa(q)=\infty$, and (8) ensures (7). See [9] for a study of the case where $\kappa(q)>0$ for every $q\geq 0$.

We shall also sometimes consider the case when the equation $\kappa(q)=0$ possesses a second solution, which we shall then denote by ω_- . More precisely, we will say that Cramér's hypothesis holds when

$$\text{there exists } \omega_-<\omega_+ \text{ such that } \kappa(\omega_-)=0 \quad \text{and} \quad \kappa'(\omega_-)>-\infty \quad (9)$$

(note that $\kappa'(\omega_-)<0$ by convexity of κ).

We are now able to introduce a first noticeable martingale.

Proposition 2.2. *Let ω be any root of the equation $\kappa(\omega)=0$. Then for every $x>0$, the process*

$$X(t)^\omega+\sum_{0<s\leq t,s<\zeta}|\Delta_-X(s)|^\omega, \quad t\geq 0,$$

(with the usual implicit convention that $X(t)^\omega=0$ whenever $t\geq\zeta$) is a uniformly integrable martingale under P_x , with terminal value $\sum_{0<s<\zeta}|\Delta_-X(s)|^\omega$.

Proof. For $q=\omega$ a root of κ , the right-hand side of (6) reduces to x^ω , and an application of the Markov property at time t yields

$$E_x\left(\sum_{0<s<\zeta}|\Delta_-X(s)|^\omega\left|\mathcal{F}_t\right.\right)=X(t)^\omega+\sum_{0<s\leq t,s<\zeta}|\Delta_-X(s)|^\omega.$$

This proves our claim. □

Even though the martingale of Proposition 2.2 shall only play a rather minor part in the rest of this paper, it is a close relative to the more important martingales which shall be introduced later on, and already points at the central role of the cumulant function κ and its roots.

2.2 Cell systems and branching random walks

We next introduce the notion of cell system and related canonical notation. We use the Ulam tree $\mathbb{U} = \bigcup_{n \geq 0} \mathbb{N}^n$ where $\mathbb{N} = \{1, 2, \dots\}$ to encode the genealogy of a family of cells which evolve and split as time passes. We define a cell system as a family $\mathcal{X} := (\mathcal{X}_u, u \in \mathbb{U})$, where each $\mathcal{X}_u = (\mathcal{X}_u(t))_{t \geq 0}$ should be thought of as the size of the cell labelled by u as a function of its *age* t . The system also implicitly encodes the birth-times b_u of those cells.

Specifically, each \mathcal{X}_u is a càdlàg trajectory with values in $(0, \infty) \cup \{\partial\}$, which fulfills the following properties:

- ∂ is an absorbing state, that is $\mathcal{X}_u(t) = \partial$ for all $t \geq \zeta_u := \inf\{t \geq 0 : \mathcal{X}_u(t) = \partial\}$,
- either $\zeta_u < \infty$ or $\lim_{t \rightarrow \infty} \mathcal{X}_u(t) = 0$.

We should think of ζ_u as the lifetime of the cell u , and stress that $\zeta_u = 0$ (that is $\mathcal{X}_u(t) \equiv \partial$) and $\zeta_u = \infty$ (that is $\mathcal{X}_u(t) \in (0, \infty)$ for all $t \geq 0$) are both allowed. The *negative* jumps of cells will play a specific part in this work. The second condition above ensures that for every given $\varepsilon > 0$, the process \mathcal{X}_u has at most finitely many negative jumps of absolute sizes greater than ε . This enables us to enumerate the sequence of the positive jump sizes and times of $-\mathcal{X}_u$ in the decreasing lexicographic order, say $(x_1, \beta_1), (x_2, \beta_2), \dots$, that is either $x_i = x_{i+1}$ and then $\beta_i > \beta_{i+1}$, or $x_i > x_{i+1}$. In the case when \mathcal{X}_u has only a finite number of negative jumps, say n , then we agree that $x_i = \partial$ and $\beta_i = \infty$ for all $i > n$. The third condition that we impose on cell systems, is that the sequence of negative jump sizes and times of a cell u encodes the birth-times and sizes at birth of its children $\{ui : i \in \mathbb{N}\}$. That is:

- for every $i \in \mathbb{N}$, the birth time b_{ui} of the cell ui is given by $b_{ui} = b_u + \beta_i$, and $\mathcal{X}_{ui}(0) = x_i$.

In words, we interpret the negative jumps of \mathcal{X}_u as birth events of the cell system, each jump of size $-x < 0$ corresponding to the birth of a new cell with initial size x , and daughter cells are enumerated in the decreasing order of the sizes at birth. Note that if \mathcal{X}_u has only finitely many negative jumps, says n , then $\mathcal{X}_{ui} \equiv \partial$ for all $i > n$.

We now introduce for every $x > 0$ a probability distribution for cell systems, denoted by \mathcal{P}_x , which is of Crump-Mode-Jagers type and can be described recursively as follows. The Eve cell, \mathcal{X}_\emptyset , has the law P_x of the self-similar Markov process X with characteristics (Ψ, α) . Given \mathcal{X}_\emptyset , the processes of the sizes of cells at the first generation, $\mathcal{X}_i = (\mathcal{X}_i(s), s \geq 0)$ for $i \in \mathbb{N}$, has the distribution of a sequence of independent processes with respective laws P_{x_i} , where $x_1 \geq x_2 \geq \dots > 0$ denotes the ranked sequence of the positive jump sizes of $-\mathcal{X}_\emptyset$. We continue in an obvious way for the second generation, and so on for the next generations; we refer to Jagers [24] for the rigorous argument showing that this indeed defines uniquely the law \mathcal{P}_x . It will be convenient for definiteness to agree that \mathcal{P}_∂ denotes the law of the degenerate process on \mathbb{U} such that $\mathcal{X}_u \equiv \partial$ for every $u \in \mathbb{U}$, $b_\emptyset = 0$ and $b_u = \infty$ for $u \neq \emptyset$. We shall also write \mathcal{E}_x for the mathematical expectation under the law \mathcal{P}_x . So, the self-similar Markov process X governs the evolution of typical cells under \mathcal{P}_x , and X will thus be often referred to as a cell process in this setting.

It is readily seen from the self-similarity of cells and the branching property that the point process on \mathbb{R} induced by the negative of the logarithms of the initial sizes of cells at a given generation,

$$\mathcal{Z}_n(dz) = \sum_{|u|=n} \delta_{-\ln \mathcal{X}_u(0)}(dz), \quad n \geq 0,$$

is a *branching random walk* (of course, the possible atoms corresponding to $\mathcal{X}_u(0) = \partial$ are discarded in the previous sum, and the same convention shall apply implicitly in the sequel). Roughly speaking, this

means that for each generation n , the point measure \mathcal{Z}_{n+1} is obtained from \mathcal{Z}_n by replacing each of its atoms, say z , by a random cloud of atoms, $\{z + y_i^z : i \in \mathbb{N}\}$, where the family $\{y_i^z : i \in \mathbb{N}\}$ has a fixed distribution, and to different atoms z of \mathcal{Z}_n correspond independent families $\{y_i^z : i \in \mathbb{N}\}$. We stress that the Lamperti transformation has no effect on the sizes of the jumps (the sizes of the jumps of X and of $x \exp(\xi)$ are obviously the same) and thus no effect either on the branching random walk \mathcal{Z}_n . In particular the law of \mathcal{Z}_n does not depend on the self-similarity parameter α .

The Laplace transform of the intensity of the point measure \mathcal{Z}_1 , which is defined by

$$m(q) := \mathcal{E}_1(\langle \mathcal{Z}_1, \exp(-q \cdot) \rangle) = \mathcal{E}_1\left(\sum_{i=1}^{\infty} \mathcal{X}_i^q(0)\right), \quad q \in \mathbb{R},$$

plays a crucial role in the study of branching random walks. In our setting, it is computed explicitly in terms of the function κ in (6) and equals

$$m(q) = 1 - \kappa(q)/\Psi(q) \quad \text{when } \Psi(q) < 0 \text{ and } \kappa(q) < \infty \quad (10)$$

and infinite otherwise. In particular, when $\kappa(q) < 0$, the structure of the branching random walk yields that

$$\mathcal{E}_1\left(\sum_{u \in \mathbb{U}} \mathcal{X}_u^q(0)\right) = \frac{\Psi(q)}{\kappa(q)}. \quad (11)$$

The Laplace transform of the intensity $m(q)$ opens the way to additive martingales, and in particular intrinsic martingales. These have a fundamental role in the study of branching random walks, in particular in connection with the celebrated spinal decomposition (see for instance the Lecture Notes by Shi [34] and references therein), and we shall especially be interested in describing its applications to self-similar growth-fragmentations.

2.3 Two genealogical martingales and the intrinsic area measure

We start this section by observing from (10) that there is the equivalence

$$\kappa(\omega) = 0 \quad \Longleftrightarrow \quad m(\omega) = 1,$$

and more precisely (recall that ω_+ is the largest root of the equation $\kappa(\omega) = 0$) we have

$$m(\omega_+) = 1 \quad \text{and} \quad m'(\omega_+) > 0.$$

Further, if Cramér's condition (9) holds and ω_- stands for the smallest root, then, in the setting of branching random walks, we have

$$m(\omega_-) = 1 \quad \text{and} \quad m'(\omega_-) \in (-\infty, 0),$$

and one refers to ω_- as the Malthusian parameter (see e.g. [13, Sec. 4]). This points at a pair of remarkable martingales, as we shall now explain.

We write \mathcal{G}_n for the sigma-field generated by the cells with generation at most n , i.e. $\mathcal{G}_n = \sigma(\mathcal{X}_u : |u| \leq n)$ and $\mathcal{G}_\infty = \bigvee_{n \geq 0} \mathcal{G}_n$. Note that for a node u at generation $|u| = n \geq 1$, the initial value $\mathcal{X}_u(0)$ of the cell labeled by u is measurable with respect to the cell \mathcal{X}_{u-} , where $u-$ denotes the parent of u at generation $n-1$. We introduce

$$\mathcal{M}^+(n) := \sum_{|u|=n+1} \mathcal{X}_u^{\omega_+}(0), \quad n \geq 0,$$

which is thus a \mathcal{G}_n -measurable variable.

Lemma 2.3. *For every $x > 0$, $(\mathcal{M}^+(n))_{n \geq 0}$ is a (\mathcal{G}_n) -martingale which converges to 0, \mathcal{P}_x -a.s.*

Proof. Recall from Sec. 2.2 that the point process \mathcal{Z}_n formed by the logarithm of the sizes at births of cells at generation n yields a branching random walk, and that $m(\omega_+) = 1$ according to (10). It follows that \mathcal{M}^+ can be viewed as a special instance of a so-called additive martingale which naturally arises in this framework, namely

$$\mathcal{M}^+(n) = \int_{\mathbb{R}} z^{-\omega_+} \mathcal{Z}_{n+1}(dz).$$

We can then use results of Biggins (see Theorem A in [13]) on branching random walks to check that its terminal value is 0 a.s. since we have $m'(\omega_+) > 0$. \square

We now assume throughout the rest of this section that the Cramér's hypothesis (9) holds, and introduce

$$\mathcal{M}^-(n) := \sum_{|u|=n+1} \mathcal{X}_u^{\omega_-}(0), \quad n \geq 0.$$

The next statement gathers some important properties of this process. In this direction, call the cell process X geometric if its trajectories, say starting from 1, take values in $\{r^z : z \in \mathbb{Z}\}$ a.s. for some fixed $r > 0$. By Lamperti's transformation, X is geometric if and only the Lévy process ξ is lattice, i.e. lives in $c\mathbb{Z}$ for some $c > 0$.

Lemma 2.4. *The following assertions hold.*

- (i) $\mathcal{M}^-(1) \in L^{\omega_+/\omega_-}(\mathcal{P}_x)$.
- (ii) $\mathcal{M}^- = (\mathcal{M}^-(n) : n \geq 0)$ is a uniformly integrable \mathcal{G}_n -martingale under \mathcal{P}_x . Further its terminal value $\mathcal{M}^-(\infty)$ is strictly positive \mathcal{P}_x -a.s. whenever $k = 0$ or $\Lambda((-\infty, 0)) = \infty$.
- (iii) Suppose that the cell process X is not geometric. We have

$$\mathcal{P}_x(\mathcal{M}^-(\infty) > t) \underset{t \rightarrow \infty}{\sim} c \cdot t^{-\omega_+/\omega_-}$$

for a certain constant $c > 0$. In particular, the r -moment of $\mathcal{M}^-(\infty)$ is finite if and only if $r < \omega_+/\omega_-$.

Proof. (i) We start by observing that

$$\int_0^\zeta X^{\alpha+q}(s)ds \in L^{\omega_+/q}(P_1) \quad \text{for every } q \in (0, \omega_+]. \quad (12)$$

Indeed, using Lamperti's transformation, there is the identity

$$\int_0^\zeta X^{\alpha+q}(s)ds = \int_0^\infty e^{q\xi(t)}dt$$

(see the calculation in the proof of Lemma 2.1), and according to Lemma 3 of Rivero [32], the exponential functional $\int_0^\infty e^{q\xi(t)}dt$ of the Lévy process ξ has indeed a finite moment of order ω_+/q , since

$$E(\exp(\xi(1)\omega_+)) = \exp(\Psi(\omega_+)) < 1.$$

Next, recall that, by construction, $\mathcal{M}^-(1)$ has the same law as $S(\zeta-)$, where

$$S(t) := \sum_{0 < s \leq t} |\Delta X(s)|^{\omega_-} \quad \text{for } t \in [0, \zeta).$$

Using the Lamperti transformation, we easily see that the predictable compensator of S is given by

$$S^{(p)}(t) := \int_0^t X^{\alpha+\omega_-}(s) \left(\int_{(-\infty,0)} (1-e^y)^{\omega_-} \Lambda(dy) \right) ds,$$

and since $\int_{(-\infty,0)} (1-e^y)^q \Lambda(dy) = \kappa(q) - \Psi(q)$, we have simply that the process

$$S(t) - S^{(p)}(t) = \sum_{0 < s \leq t} |\Delta_- X(s)|^{\omega_-} + \Psi(\omega_-) \int_0^t X^{\alpha+\omega_-}(s) ds$$

is a martingale. This martingale is obviously purely discontinuous and has quadratic variation

$$\sum_{0 < s \leq t} |\Delta_- X(s)|^{2\omega_-}.$$

Using (12), we know that $S^{(p)}(\zeta-) \in L^{\omega_+/\omega_-}(P_1)$, and then the Burkholder-Davis-Gundy inequality reduces the proof to checking that

$$\sum_{0 < s < \zeta} |\Delta_- X(s)|^{2\omega_-} \in L^{\omega_+/2\omega_-}(P_1). \quad (13)$$

Suppose first $\omega_+/\omega_- \leq 2$, so that

$$\left(\sum_{0 < s < \zeta} |\Delta_- X(s)|^{2\omega_-} \right)^{\omega_+/2\omega_-} \leq \sum_{0 < s < \zeta} |\Delta_- X(s)|^{\omega_+}.$$

Since $\Psi(\omega_+) < \kappa(\omega_+) = 0$, we know further from Lemma 2.1 that

$$E_1 \left(\sum_{0 < s < \zeta} |\Delta_- X(s)|^{\omega_+} \right) = -\Psi(\omega_+) E_1 \left(\int_0^\zeta X^{\alpha+\omega_+}(s) ds \right) = 1,$$

which proves (13).

Suppose next that $2 < \omega_+/\omega_- \leq 4$. The very same argument as above, with $2\omega_-$ replacing ω_- , shows that

$$\sum_{0 < s \leq t} |\Delta_- X(s)|^{2\omega_-} + \Psi(2\omega_-) \int_0^t X^{\alpha+2\omega_-}(s) ds$$

is a purely discontinuous martingale with quadratic variation $\sum_{0 < s \leq t} |\Delta_- X(s)|^{4\omega_-}$, and we conclude in the same way that (13) holds. We can repeat the argument when $2^k < \omega_+/\omega_- \leq 2^{k+1}$ for every $k \in \mathbb{N}$, and thus complete the proof of our claim by iteration.

(ii) Recall that $m(\omega_-) = 1$ and $m'(\omega_-) < 0$, so our first assertion follows again from Theorem A of Biggins [13] (note that the $L \log L$ condition is fulfilled, thanks to (i)). When further $k = 0$ or $\Lambda((-\infty, 0)) = \infty$, each cell begets at least one daughter before dying a.s., and the branching random walk \mathcal{Z}_n is never extinct a.s. It then follows from the branching property that $\mathcal{P}_x(\mathcal{M}^-(\infty) = 0) = 0$.

(iii) This is a consequence of Theorem 2.2 of Liu [29]. Specifically, we have

$$\mathcal{E}_1 \left(\sum_{i=1}^{\infty} (\mathcal{X}_i^{\omega_-}(0))^{\omega_+/\omega_-} \right) = \mathcal{E}_1 \left(\sum_{i=1}^{\infty} \mathcal{X}_i^{\omega_+}(0) \right) = m(\omega_+) = 1,$$

and also

$$\mathcal{E}_1 \left(\sum_{i=1}^{\infty} (\mathcal{X}_i^{\omega_+}(0))^q \right) = m(q\omega_+) < \infty,$$

for every $q > 1$ with $\Psi(q\omega_+) < 0$ (note that $\Psi(\omega_+) < \kappa(\omega_+) = 0$, so there exists such $q > 1$). Finally, the fact that $\mathcal{M}^-(1) \in L^{\omega_+/\omega_-}(\mathcal{P}_x)$ has been established in (i). \square

The martingale \mathcal{M}^- is known as the *Malthusian martingale* or also the *intrinsic martingale* in the folklore of branching random walks. We point out that, by the construction of the cell system, its terminal value fulfills the following identity in distribution:

$$\mathcal{M}^-(\infty) \stackrel{(d)}{=} \sum_{i=1}^{\infty} \Delta_i^{\omega_-} \mathcal{M}_i^-(\infty), \quad (14)$$

where in the right-hand side, $(\Delta_i)_{i \in \mathbb{N}}$ denotes the sequence of the absolute values of the negative jump sizes of the self-similar process with characteristics (Ψ, α) , and $(\mathcal{M}_i^-(\infty))_{i \in \mathbb{N}}$ is a sequence of i.i.d. copies of $\mathcal{M}^-(\infty)$ which is further assumed to be independent of $(\Delta_i)_{i \in \mathbb{N}}$. In words, the distribution of the intrinsic area is a fixed point for a smoothing transformation of the type considered by many authors, and we infer in particular from results due to Biggins (see Sec. 2 in [13]), that this equation has a unique solution with given mean. It turns out that the law of $\mathcal{M}^-(\infty)$ only depends on κ (see Remark 3.11).

The terminal value $\mathcal{M}^-(\infty)$ will be referred here to as the *intrinsic area* of the associated growth-fragmentation. In Sec. 5, we will introduce a distinguished family of self-similar growth-fragmentations and identify the law of the limit of their Malthusian martingales as being size-biased versions of stable random variable (see Corollary 6.9 below). To this end, we will crucially rely on a connection between these particular growth-fragmentations and random maps where in this context $\mathcal{M}^-(\infty)$ is interpreted as the limit law for the *area* (i.e. the number of vertices) of the maps. Hence the name *intrinsic area*.

The intrinsic area $\mathcal{M}^-(\infty)$ appears in a variety of limit theorems. Here, we shall merely illustrate this in the following situation. Fix $\varepsilon > 0$ and imagine that we freeze every cell when its size becomes $< \varepsilon$ (which may occur at birth), in the sense that a frozen cell no longer evolves and in particular ceases to give birth. We write \mathbf{M}_ε^- for the sum of the sizes of frozen cells raised to the power ω_- , and claim:

Proposition 2.5. *The process $(\mathbf{M}_\varepsilon^- : 0 < \varepsilon \leq x)$ is a backward martingale under \mathcal{P}_x , and*

$$\lim_{\varepsilon \rightarrow 0+} \mathbf{M}_\varepsilon^- = \mathcal{M}^-(\infty) \quad \text{a.s. and in } L^1(\mathcal{P}_x).$$

Proof. Introduce the sigma field \mathcal{H}_ε generated the frozen system (i.e. by the cells observed until their sizes are less than ε), so $(\mathcal{H}_\varepsilon)_{\varepsilon > 0}$ is a backward filtration. We consider first just the Eve cell, and write $T_\emptyset(\varepsilon) = \inf\{t \geq 0 : \mathcal{X}_\emptyset(t) < \varepsilon\}$ and

$$\mathbf{M}_\varepsilon^-(1) = \mathcal{X}_\emptyset^{\omega_-}(T_\emptyset(\varepsilon)) + \sum_{0 < s \leq T_\emptyset(\varepsilon)} |\Delta \cdot \mathcal{X}_\emptyset(s)|^{\omega_-}.$$

We see from Proposition 2.2 and Doob's optional sampling theorem that under \mathcal{P}_x , the process $(\mathbf{M}_\varepsilon^-(1) : 0 < \varepsilon \leq x)$ is a backward martingale, which is uniformly integrable and its terminal value is $\lim_{\varepsilon \rightarrow 0+} \mathbf{M}_\varepsilon^-(1) = \mathcal{M}^-(1)$.

More generally for $n \geq 2$, imagine that we freeze each cell not only when its size becomes $< \varepsilon$, but also at birth if it has generation n . Write $\mathbf{M}_\varepsilon^-(n)$ for the sum of the sizes of those frozen cells raised to the power ω_- . The branching property of cell systems enables us to iterate the argument above, and we get that $(\mathbf{M}_\varepsilon^-(n) : 0 < \varepsilon \leq x)$ is a uniformly integrable backward $(\mathcal{P}_x, \mathcal{H}_\varepsilon)$ -martingale with terminal value $\mathcal{M}^-(n)$. Lemma 2.4 enables us to pass to the limit as $n \rightarrow \infty$, and thus

$$\mathcal{E}_x(\mathcal{M}^-(\infty) \mid \mathcal{H}_\varepsilon) = \lim_{n \rightarrow \infty} \mathbf{M}_\varepsilon^-(n)$$

is a uniformly integrable backward $(\mathcal{P}_x, \mathcal{H}_\varepsilon)$ -martingale. On the other hand, we know from Corollary 4 in [5] that \mathcal{P}_x -a.s., only finitely many cells have a maximal size greater than ε , so $\mathbf{M}_\varepsilon^-(n) = \mathbf{M}_\varepsilon^-$ for n sufficiently large, and in particular $\lim_{n \rightarrow \infty} \mathbf{M}_\varepsilon^-(n) = \mathbf{M}_\varepsilon^-$. To complete the proof, we simply observe that $\bigvee_{\varepsilon > 0} \mathcal{H}_\varepsilon = \bigvee_{n \geq 1} \mathcal{G}_n$, obviously. \square

It is further well-known that the Malthusian martingale yields an important random measure on the boundary $\partial\mathbb{U}$ of the Ulam tree; see e.g. Liu [29] for background and references. Specifically, $\partial\mathbb{U}$ is a complete metric space when endowed with the distance $d(\ell, \ell') = \exp(-\inf\{n \geq 0 : \ell(n) = \ell'(n)\})$, where the notation $\ell(n)$ designates the ancestor of the leaf ℓ at generation n . We can construct a (unique) random measure \mathcal{A} on $\partial\mathbb{U}$, which we call the *intrinsic area measure*, such that the following holds. For every $u \in \mathbb{U}$, we write $B(u) := \{\ell \in \partial\mathbb{U} : \ell(n) = u\}$ for the ball in $\partial\mathbb{U}$ which stems from u , and introduce

$$\mathcal{A}(B(u)) := \lim_{n \rightarrow \infty} \sum_{|v|=n, v(i)=u} \mathcal{X}_v^{\omega_-}(0),$$

where $i = |u|$ stands for the generation of u , and for every vertex v with $|v| \geq i$, $v(i)$ stands for the ancestor of v at generation i (the existence of this limit is ensured by the branching property). In particular, for $u = \emptyset$, the total mass of \mathcal{A} is $\mathcal{A}(\partial\mathbb{U}) = \mathcal{A}(B(\emptyset)) = \mathcal{M}^-(\infty)$.

3 Self-similar growth-fragmentations and a many-to-one formula

3.1 Self-similar growth-fragmentations

The cell system \mathcal{X} being constructed, we next define the cell population at time $t \geq 0$ as the family of the sizes of the cells alive at time t , viz.

$$\mathbf{X}(t) = \{\mathcal{X}_u(t - b_u) : u \in \mathbb{U}, b_u \leq t < b_u + \zeta_u\},$$

where b_u and ζ_u denote respectively the birth time and the lifetime of the cell u and the notation $\{\dots\}$ refers to multiset (i.e. elements are repeated according to their multiplicities). According to Theorem 2 in [5] (again, the assumption of absence of positive jumps which was made there plays no role), (8) ensures that the elements of $\mathbf{X}(t)$ can be ranked in the non-increasing order and then form a null-sequence (i.e. tending to 0), say $X_1(t) \geq X_2(t) \geq \dots \geq 0$; if $\mathbf{X}(t)$ has only finitely many elements, say n , then we agree that $X_{n+1}(t) = X_{n+2}(t) = \dots = 0$ for the sake of definiteness. Letting the time parameter t vary, we call the process of cell populations $(\mathbf{X}(t), t \geq 0)$ a *growth-fragmentation process* associated with the cell process X . We denote by \mathbb{P}_x its law under \mathcal{P}_x .

Remark 3.1. By the above construction, the law of \mathbf{X} only depends on the law of the Eve cell which in turn is characterized by (Ψ, α) . However, different laws for the Eve cell may yield to the same growth-fragmentation process \mathbf{X} . This has been analyzed in depth in [33] where it is proved that the law of \mathbf{X} is in fact characterized by the pair (κ, α) . This reference only treats the case where positive jumps are not allowed, but the same arguments apply.

We now discuss the branching property of self-similar growth-fragmentations. In this direction, we first introduce $(\mathcal{F}_t)_{t \geq 0}$, the natural filtration generated by $(\mathbf{X}(t) : t \geq 0)$, and recall from Theorem 2 in [5] that the growth-fragmentation process is Markovian with semigroup fulfilling the branching property. That is, conditionally on $\mathbf{X}(t) = (x_1, x_2, \dots)$, the shifted process $(\mathbf{X}(t+s) : s \geq 0)$ is independent of \mathcal{F}_t and its distribution is the same as that of the process obtained by taking the union (in the sense of multisets) of a sequence of independent growth-fragmentation processes with respective distributions $\mathbb{P}_{x_1}, \mathbb{P}_{x_2}, \dots$. We stress that, in general, the genealogical structure of the cell system cannot be recovered from the growth-fragmentation process \mathbf{X} alone, and this motivates working with the following enriched version.

We introduce

$$\bar{\mathbf{X}}(t) = \{(\mathcal{X}_u(t - b_u), |u|) : u \in \mathbb{U}, b_u \leq t < b_u + \zeta_u\}, \quad t \geq 0$$

that is we record not just the family of the sizes of the cells alive at time t , but also their generations. We denote the natural filtration generated by $(\bar{\mathbf{X}}(t) : t \geq 0)$ by $(\bar{\mathcal{F}}_t)_{t \geq 0}$. In this enriched setting, it should be intuitively clear that the following slight variation of the branching property still holds. A rigorous proof can be given following an argument similar to that for Proposition 2 in [5]; details are omitted.

Lemma 3.2. *For every $t \geq 0$, conditionally on $\bar{\mathbf{X}}(t) = \{(x_i, n_i) : i \in \mathbb{N}\}$, the shifted process $(\bar{\mathbf{X}}(t + s) : s \geq 0)$ is independent of $\bar{\mathcal{F}}_t$ and its distribution is the same that of the process*

$$\bigsqcup_{i \in \mathbb{N}} \bar{\mathbf{X}}_i(s) \circ \theta_{n_i}, \quad s \geq 0,$$

where \bigsqcup denotes the union in the sense of multisets, θ_n the operator that consists in shifting all generations by n , viz. $\{(y_j, k_j) : j \in \mathbb{N}\} \circ \theta_n = \{(y_j, k_j + n) : j \in \mathbb{N}\}$, and the $\bar{\mathbf{X}}_i$ are independent processes, each having the same law as $\bar{\mathbf{X}}$ under \mathcal{P}_{x_i} .

It is easy to see that \mathcal{P}_x -a.s., two different cells never split at the same time, that is for every $u, v \in \mathbb{U}$ with $u \neq v$, there is no $t \geq 0$ with $b_u \leq t, b_v \leq t$ such that both $\Delta \mathcal{X}_u(t - b_u) < 0$ and $\Delta \mathcal{X}_v(t - b_v) < 0$. Further, there is an obvious correspondence between the negative jumps of cell-processes and those of $\bar{\mathbf{X}}$. Specifically, if for some $u \in \mathbb{U}$ and $t > b_u$, $\mathcal{X}_u((t - b_u)-) = a$ and $\mathcal{X}_u(t - b_u) = a' < a$, then at time t , one element $(a, |u|)$ is removed from $\bar{\mathbf{X}}(t-)$ and is replaced by a two elements $(a', |u|)$ and $(a - a', |u| + 1)$. In the converse direction, the Eve cell process \mathcal{X}_\emptyset is recovered from $\bar{\mathbf{X}}$ by following the single term with generation 0, and this yields the birth times and sizes at birth of the cells at the first generation. Next, following the negative jumps of the elements in $\bar{\mathbf{X}}$ at the first generation, then the second generation, and so on, we can recover the family of the birth times and sizes at birth of cells at any given generation from the observation of the process $\bar{\mathbf{X}}$.

3.2 The intensity measure of a self-similar growth-fragmentation

The main object of interest in the present section is the intensity measure μ_t^x of $\mathbf{X}(t)$ under \mathbb{P}_x , which is defined by

$$\langle \mu_t^x, f \rangle := \mathbb{E}_x \left(\sum_{i=1}^{\infty} f(X_i(t)) \right),$$

with, as usual, $f : (0, \infty) \rightarrow [0, \infty)$ a generic measurable function and the convention $f(0) = 0$. We shall obtain an explicit expression for μ_t^x in terms of the transition kernel of a certain self-similar Markov process. Formulas of this type are often referred to as *many-to-one* in the literature, to stress that the intensity of a random point measure is expressed in terms of the distribution of a single particle. They play a fundamental role in the analysis of branching type processes; their interpretation and usefulness have been revealed in the pioneer article by Lyons, Pemantle & Peres [30] (see also the Lecture Notes by Shi [34]). We stress however that, even though self-similar growth-fragmentation processes are Markovian processes which fulfill the branching property (see Theorem 2 in [5]), their construction based on cell systems \mathcal{X} and their genealogical structures of Crump-Mode-Jagers type make the analysis of $(\mathbf{X}(t))_{t \geq 0}$ as a process evolving with time t rather un-direct, and the classical methods for establishing many-to-one formulas for branching processes do not apply straightforwardly in our setting.

The following statement is the key to the analysis of the intensity measure μ_t^x .

Proposition 3.3. *For every q such that $\kappa(q) < 0$, we have*

$$\mathbb{E}_1 \left(\int_0^\infty \left(\sum_{i=1}^\infty X_i^{q+\alpha}(t) \right) dt \right) = -1/\kappa(q).$$

Proof. We start by observing that, due to the very construction of the growth-fragmentation process \mathbf{X} , there is the identity

$$\int_0^\infty \left(\sum_{i=1}^\infty X_i^{q+\alpha}(t) \right) dt = \sum_{u \in \mathbb{U}} \int_0^{\zeta_u} \mathcal{X}_u^{q+\alpha}(t) dt.$$

Because for each $u \in \mathbb{U}$, conditionally on $\mathcal{X}_u(0) = x$, the process \mathcal{X}_u of the size of the cell labelled by u has the distribution of the self-similar Markov process X with characteristics (Ψ, α) started from x , we deduce from Lemma 2.1 that

$$\mathbb{E}_1 \left(\int_0^{\zeta_u} \mathcal{X}_u^{q+\alpha}(t) dt \right) = \mathbb{E}_1 \left(-\frac{\mathcal{X}_u(0)^q}{\Psi(q)} \right).$$

An appeal to (11) completes the proof. \square

In order to state the main result of this section, recall that ω_+ is the largest root of the equation $\kappa(q) = 0$ and fulfills (7). We define

$$\Phi^+(q) := \kappa(q + \omega_+)$$

and record for future use the following elementary facts (see Lemma 3.1 in [10] for closely related calculations).

Lemma 3.4. *The following assertions hold.*

- (i) *The function Φ^+ can be expressed in a Lévy-Khintchin form similar to (1). More precisely, the killing rate is 0, the Gaussian coefficient $\frac{1}{2}\sigma^2$, and the Lévy measure Π given by*

$$\Pi(dx) := e^{x\omega_+} \left(\Lambda(dx) + \tilde{\Lambda}(dx) \right), \quad x \in \mathbb{R},$$

where $\tilde{\Lambda}(dx)$ denotes the image of $\mathbb{1}_{x < 0} \Lambda(dx)$ by the map $x \mapsto \ln(1 - e^x)$.

- (ii) *Therefore Φ^+ is the Laplace exponent of a Lévy process, say $\eta^+ = (\eta^+(t), t \geq 0)$. Further $\Phi^+(0) = 0$, $(\Phi^+)'(0) = \kappa'(\omega_+) > 0$, and thus η^+ drifts to $+\infty$.*

With the notation of Sec. 2.1 we write $Y^+ = (Y^+(t) : t \geq 0)$ for the self-similar Markov process with characteristics (Φ^+, α) we denote P_x^+ for its law started from $Y^+(0) = x$, and then $\rho_t^+(x, dy)$ for its transition kernel, viz.

$$\langle \rho_t^+(x, \cdot), f \rangle = E_x^+ (f(Y^+(t))).$$

We are now able to claim the following.

Theorem 3.5. *For every $x > 0$ and $t \geq 0$, there is the identity*

$$\mu_t^x(dy) = \left(\frac{x}{y} \right)^{\omega_+} \rho_t^+(x, dy), \quad y > 0.$$

The proof of Theorem 3.5 relies on a kind of a converse to Lemma 2.1, for which we must first introduce some notation. Consider a sub-Markovian transition kernel $(\varrho_t(x, dy))_{t \geq 0}$ on $(0, \infty)$; as usual this kernel can be viewed as Markovian by the introduction of the cemetery point ∂ . That is, each $\varrho_t(x, dy)$ is a sub-probability measure on $(0, \infty)$ which depends in a measurable way on the variable x , and the Chapman-Kolmogorov equation

$$\langle \varrho_{t+s}(x, \cdot), f \rangle := \int_0^\infty f(y) \varrho_{t+s}(x, dy) = \int_0^\infty \langle \varrho_t(z, \cdot), f \rangle \varrho_s(x, dz)$$

holds for every $s, t \geq 0$. We assume further that this transition kernel is self-similar; specifically, for every $t \geq 0$ and $x > 0$, $\varrho_t(x, \cdot)$ is the image of $\varrho_{tx^\alpha}(1, dy)$ by the map $y \mapsto xy$, that is

$$\langle \varrho_t(x, \cdot), f \rangle = \langle \varrho_{tx^\alpha}(1, \cdot), f(x \cdot) \rangle.$$

Recall that then $t \mapsto \varrho_t(x, dy)$ is continuous for the topology of vague convergence, see Lemma 2.1 in [27].

Lemma 3.6. *Under the assumption above, suppose further that the potential measure induced by $(\varrho_t)_{t \geq 0}$ has the same Mellin transform as U , that is*

$$\int_0^\infty \int_{(0, \infty)} y^{q+\alpha} \varrho_t(1, dy) dt = -\frac{1}{\Psi(q)} \quad \text{whenever } \Psi(q) < 0.$$

Then $(\varrho_t(x, dy))_{t \geq 0}$ is the transition kernel of the self-similar Markov process X with characteristics (Ψ, α) .

Proof. Our assumptions ensure that $(\varrho_t(x, dy))_{t \geq 0}$ is the transition kernel of a self-similar Markov process on $(0, \infty)$, say $Y = (Y(t), t \geq 0)$. We know from Lamperti [27] that there is a (possibly killed) Lévy process $\eta = (\eta(t))_{t \geq 0}$ whose Lamperti transform has the same law as Y . We write Φ for the Laplace exponent of η , which is given by $\Phi(q) = \ln \mathbb{E}(\exp(q\eta(1)))$.

Lemma 2.1 shows that $\Phi(q) = \Psi(q)$ provided that $\Psi(q) < 0$. Since the set $\{q > 0 : \Psi(q) < 0\}$ has a non-empty interior, we conclude by analytic continuation of the characteristic functions that η has the same law as ξ . \square

We now have all the ingredients to establish Theorem 3.5.

Proof of Theorem 3.5. We first pick any $\theta > 0$ with $\kappa(\theta) < 0$ (note that this forces $\theta < \omega_+$) and consider $\tilde{\Phi}(q) := \kappa(\theta + q)$. Then $\tilde{\Phi}$ is also the Laplace exponent of a Lévy process, say $\tilde{\eta}$, which has killing rate $-\kappa(\theta) > 0$. We write \tilde{Y} for the self-similar Markov process with characteristics $(\tilde{\Phi}, \alpha)$, and $\tilde{\rho}_t(x, dy)$ for its transition (sub-)probabilities. As $\tilde{\Phi}(q) = \Phi^+(q + \theta - \omega_+)$, the laws of the processes η^+ and $\tilde{\eta}$ are equivalent on every finite horizon, and more precisely, there is the absolute continuity relation

$$E(F(\tilde{\eta}(s) : 0 \leq s \leq t)) = E(F(\eta^+(s) : 0 \leq s \leq t) \exp((\theta - \omega_+)\eta^+(t))),$$

for every functional $F \geq 0$ which is zero when applied to a path with lifetime less than t . It is straightforward to deduce that there is the identity

$$\tilde{\rho}_t(x, dy) = \left(\frac{y}{x}\right)^{\theta - \omega_+} \rho_t^+(x, dy),$$

and therefore our statement can also be rephrased as

$$\mu_t^x(dy) = \left(\frac{x}{y}\right)^\theta \tilde{\rho}_t(x, dy), \quad y > 0. \tag{15}$$

To prove (15), we first note from the (temporal) branching property of growth-fragmentation processes (see Theorem 2 in [5], or Lemma 3.2 here) that the family $(\mu_t^x)_{x,t \geq 0}$ fulfills the Chapman-Kolmogorov identity

$$\langle \mu_{t+s}^x, f \rangle = \int_{(0,\infty)} \langle \mu_t^y, f \rangle \mu_s(x, dy).$$

However, this is not a Markovian transition kernel, as $\langle \mu_t^y, 1 \rangle$ may be larger than 1, and even infinite. Nonetheless, thanks to Theorem 2 in [5], since $\kappa(\theta) < 0$, the power function $h : y \mapsto y^\theta$ is excessive for the growth-fragmentation process \mathbf{X} , that is $\langle \mu_t^x, h \rangle \leq h(x)$ for every $x > 0$. It follows that if we introduce the super-harmonic transform

$$\tilde{\varrho}_t(x, dy) := \left(\frac{y}{x}\right)^\theta \mu_t^x(dy),$$

then the semigroup property of the measures $\mu_t^x(dy)$ is transmitted to $\tilde{\varrho}_t(x, dy)$, which then form a transition (sub-)probability kernel of a Markov process on $(0, \infty)$.

Further, the measures $\mu_t^x(dy)$ fulfill the scaling property

$$\langle \mu_t^x, f \rangle = \langle \mu_{tx}^1, f(x \cdot) \rangle$$

(see again Theorem 2 in [5]), and this also propagates to $\tilde{\varrho}_t(x, dy)$.

Now it suffices to observe that for every $q > 0$ with $\kappa(q) < 0$, we have

$$\int_0^\infty \int_{(0,\infty)} y^{q+\alpha-\theta} \tilde{\varrho}_t(1, dy) = \int_0^\infty \int_{(0,\infty)} y^{q+\alpha} \mu_t^1(dy) = -\frac{1}{\kappa(q)},$$

where the second identity follows from Proposition 3.3. Recall that $\kappa(q) = \tilde{\Phi}(q - \theta)$ and Lemma 3.6 to conclude that $\tilde{\varrho}$ is the transition kernel of the self-similar Markov process with characteristics $(\tilde{\Phi}, \alpha)$, that is $\tilde{\varrho} = \tilde{\rho}$. \square

3.3 Two temporal martingales

We shall now present some applications of Theorem 3.5, starting with remarkable temporal martingales (recall also Proposition 2.2) which, roughly speaking, are the temporal versions of the genealogical martingales of Sec. 2.3.

Corollary 3.7. *The following assertions hold:*

(i) *For every $x > 0$, the process*

$$M^+(t) := \sum_{i=1}^{\infty} X_i^{\omega^+}(t), \quad t \geq 0$$

is a \mathbb{P}_x -martingale if $\alpha \leq 0$, and a \mathbb{P}_x -supermartingale which converges to 0 in $L^1(\mathbb{P}_x)$ if $\alpha > 0$.

(ii) *Further, for every real number q with $\kappa(q) < 0$, $\sum_{i=1}^{\infty} X_i^q(t)$ is \mathbb{P}_x -supermartingale which converges to 0 in $L^1(\mathbb{P}_x)$.*

Corollary 3.7 can also be phrased as follows in the setting of potential theory for Markov processes. We view \mathbf{X} as a process with values in the space of non-increasing null sequences $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$, and for every $q > 0$, we consider the function defined by

$$F_q(\mathbf{x}) = \sum_{n=1}^{\infty} x_n^q,$$

so that $M^+(t) = F_{\omega^+}(\mathbf{X}(t))$. Then for $\alpha \leq 0$, F_{ω^+} is invariant for the growth-fragmentation \mathbf{X} , whereas for $\alpha > 0$, F_{ω^+} is purely excessive.

Proof. (i) It is convenient to use the notation above. We know from Theorem 3.5 that for every $x > 0$, there are the identities

$$\mathbb{E}_x(F_{\omega_+}(\mathbf{X}(t))) = \langle \mu_t^x, F_{\omega_+} \rangle = x^{\omega_+} \rho_t^+(x, (0, \infty)) = x^{\omega_+} P(Y^+(t) \in (0, \infty) \mid Y^+(0) = x),$$

where Y^+ denotes the self-similar Markov process with characteristics (Φ^+, α) . Recall that $\Phi^+(0) = 0$ and $(\Phi^+)'(0) = \kappa'(\omega_+) > 0$, so that the Lévy process η with Laplace exponent Φ^+ drifts to $+\infty$. By Lamperti's construction, this entails that the lifetime of Y^+ is a.s. infinite if $\alpha \leq 0$, and a.s. finite if $\alpha > 0$. Thus $\mathbb{E}_x(F_{\omega_+}(\mathbf{X}(t))) = x^{\omega_+}$ if $\alpha \leq 0$, whereas $\lim_{t \rightarrow \infty} \mathbb{E}_x(F_{\omega_+}(\mathbf{X}(t))) = 0$ if $\alpha > 0$, and our claim follows easily from the branching property of growth-fragmentations.

(ii) Recall that when $\kappa(\theta) < 0$, the function $\tilde{\Phi}(q) := \kappa(q + \theta)$ is the Laplace exponent of another Lévy process, say $\tilde{\eta}$, which has killing rate $\tilde{k} = -\kappa(\theta) > 0$. Therefore the self-similar Markov process with characteristics $(\tilde{\Phi}, \alpha)$ has a finite lifetime a.s., and if we write $\tilde{\rho}_t(x, dy)$ for its transition kernel, we have $\lim_{t \rightarrow \infty} \tilde{\rho}_t(x, (0, \infty)) = 0$. We just need to recall from (15) that

$$\mathbb{E}_x(F_\theta(\mathbf{X}(t))) = \langle \mu_t^x, F_\theta \rangle = x^\theta \tilde{\rho}_t(x, (0, \infty)),$$

our second assertion is proved. \square

We now relate the (super-)martingale in continuous time, $M^+(t)$, to the discrete parameter martingale, $\mathcal{M}^+(n)$, which was introduced in the preceding section. In this direction, recall that $\bar{\mathbf{X}}(t)$ denotes the enriched growth fragmentation process in which the sizes of cells alive at time t are recorded together with their generations, and that $(\bar{\mathcal{F}}_t)_{t \geq 0}$ denotes the natural filtration of this enriched process.

Lemma 3.8. *For every $t \geq 0$, there is the identity*

$$M^+(t) = \lim_{n \rightarrow \infty} \mathcal{E}_x(\mathcal{M}^+(n) \mid \bar{\mathcal{F}}_t) \quad \mathcal{P}_x\text{-a.s.},$$

and this convergence also holds in $L^1(\mathcal{P}_x)$ when $\alpha \leq 0$.

Proof. We first claim that for every $n \geq 0$ and $t \geq 0$, we have

$$\mathcal{E}_x(\mathcal{M}^+(n) \mid \bar{\mathcal{F}}_t) = \sum_{|u|=n+1} \mathbf{1}_{b_u \leq t} \mathcal{X}_u^{\omega_+}(0) + \sum_{|v| \leq n} \mathbf{1}_{b_v \leq t} \mathcal{X}_v^{\omega_+}(t - b_v), \quad \mathcal{P}_x\text{-a.s.}$$

Indeed, we decompose

$$\mathcal{M}^+(n) = \sum_{|u|=n+1} \mathbf{1}_{b_u \leq t} \mathcal{X}_u^{\omega_+}(0) + \sum_{|u|=n+1} \mathbf{1}_{b_u > t} \mathcal{X}_u^{\omega_+}(0),$$

and note that the first term in the sum of the right-hand side is $\bar{\mathcal{F}}_t$ -measurable; see the comments after Lemma 3.2. For the second term, we observe that the set of cells at generation $n+1$ which are born after time t admits a natural partition into subfamilies of cells having the same most recent ancestor alive at time t . Applications of the branching property stated in Lemma 3.2 and of the martingale property of \mathcal{M}^+ then show that

$$\mathcal{E}_x \left(\sum_{|u|=n+1} \mathbf{1}_{b_u > t} \mathcal{X}_u^{\omega_+}(0) \mid \bar{\mathcal{F}}_t \right) = \sum_{|v| \leq n} \mathbf{1}_{b_v \leq t} \mathcal{X}_v^{\omega_+}(t - b_v)$$

(note that the right-hand side is actually $\bar{\mathbf{X}}(t)$ -measurable), which establishes our claim.

Because the martingale $\mathcal{M}^+(n)$ converges to 0 a.s., we have also

$$\lim_{n \rightarrow \infty} \sum_{|u|=n+1} \mathbf{1}_{b_u \leq t} \mathcal{X}_u^{\omega+}(0) = 0.$$

On the other hand, by monotone convergence, we have

$$\lim_{n \rightarrow \infty} \sum_{|v| \leq n} \mathbf{1}_{b_v \leq t} \mathcal{X}_v^{\omega+}(t - b_v) = \sum_{v \in \mathbb{U}} \mathbf{1}_{b_v \leq t} \mathcal{X}_v^{\omega+}(t - b_v) = M^+(t).$$

This proves the first assertion of the statement, and the second then follows from the fact that M^+ is a martingale when $\alpha \leq 0$, so

$$\mathcal{E}_x(M^+(t)) = x^{\omega+} = \mathcal{E}_x(\mathcal{M}^+(n)).$$

We can then complete the proof with an application of Scheffé's lemma. \square

We next assume throughout the rest of this section that Cramér's hypothesis (9) is fulfilled, and introduce similarly the process

$$M^-(t) := \sum_{i=1}^{\infty} X_i^{\omega-}(t), \quad t \geq 0.$$

Corollary 3.9. *Suppose that (9) holds. Then*

(i) *If $\alpha \geq 0$, then M^- is a \mathbb{P}_x -martingale for all $x > 0$, whereas if $\alpha < 0$, then M^- is a \mathbb{P}_x -supermartingale which converges to 0 in $L^1(\mathbb{P}_x)$.*

(ii) *More precisely, if $\alpha < 0$, then*

$$\mathbb{E}_x(M^-(t)) \sim cx^{\omega_+ t^{(\omega_+ - \omega_-)/\alpha}} \quad \text{as } t \rightarrow \infty,$$

for some constant $c \in (0, \infty)$.

(iii) *For $\alpha > 0$, then we have also*

$$\mathbb{E}_x(M^+(t)) \sim c'x^{\omega_- t^{-(\omega_+ - \omega_-)/\alpha}} \quad \text{as } t \rightarrow \infty,$$

for some constant $c' \in (0, \infty)$.

Proof. (i) When (9) holds, we can introduce $\Phi^-(q) := \kappa(q + \omega_-) = \Phi^+(q + \omega_- - \omega_+)$, which is the Laplace exponent of another Lévy process, say η^- , which has no killing and drifts to $-\infty$. If we write $\rho_t^-(x, dy)$ for the transition kernel of the self-similar Markov process with characteristics (Φ^-, α) , the same argument as in the proof of Corollary 3.7(ii) shows that $\rho_t^-(x, (0, \infty)) \equiv 1$ when $\alpha \geq 0$, whereas $\lim_{t \rightarrow \infty} \rho_t^-(x, (0, \infty)) = 0$ when $\alpha < 0$, and that

$$\mathbb{E}_x(F_{\omega_-}(\mathbf{X}(t))) = \langle \mu_t^x, F_{\omega_-} \rangle = x^{\omega_-} \rho_t^-(x, (0, \infty)),$$

which entails our first assertion.

(ii) Assume $\alpha < 0$ and take first $x = 1$. Then Lamperti's construction shows that

$$\rho_t^-(1, (0, \infty)) = P\left(\int_0^\infty \exp(-\alpha \eta^-(s)) ds > t\right),$$

and the right-hand side can be estimated using results by Rivero [32]. Indeed, the Lévy process η^- (which drifts to $-\infty$) fulfills Cramér's condition, namely

$$E(\exp((\omega_+ - \omega_-)\eta^-(1))) = 1 \quad \text{and} \quad E(|\eta^-(1)| \exp((\omega_+ - \omega_-)\eta^-(1))) = E(|\eta^+(1)|) < \infty,$$

and it follows from Lemma 4 in [32] that

$$P\left(\int_0^\infty \exp(-\alpha\eta^-(s))ds > t\right) \sim ct^{(\omega_+ - \omega_-)/\alpha} \quad \text{as } t \rightarrow \infty.$$

This establishes our claim for $x = 1$, and the general case then follows by scaling.

(iii) *Mutatis mutandis*, the proof is the same as for (ii). \square

We point out that the same calculation as above, but replacing the use of Lemma 4 in [32] by Theorem 5 in [1], enables us to extend Corollary 3.9(ii) as follows. Assuming still $\alpha < 0$, but without requiring (9) any longer, we consider any $q > 0$ with $\kappa(q) \leq 0$. Then we have

$$\mathbb{E}_x\left(\sum_{i=1}^\infty X_i^q(t)\right) \sim c(q)x^{\omega_+}t^{(\omega_+ - q)/\alpha} \quad \text{as } t \rightarrow \infty, \quad (16)$$

for some constant $c(q) \in (0, \infty)$. Similarly, for $\alpha > 0$ and now requiring (9) again, we have for every $q > 0$ with $\kappa(q) < 0$ that

$$\mathbb{E}_x\left(\sum_{i=1}^\infty X_i^q(t)\right) \sim c'(q)x^{\omega_-}t^{-(\omega_- - q)/\alpha} \quad \text{as } t \rightarrow \infty.$$

The intrinsic area measure \mathcal{A} which was introduced at the end of Sec. 2.3 enables us to describe close links between the Malthusian martingale \mathcal{M}^- , its terminal value, and the (super)-martingale M^- . Recall that $(\bar{\mathcal{F}}_t)_{t \geq 0}$ denotes the canonical filtration of the enriched growth-fragmentation process $\bar{\mathbf{X}}$ in which generations are recorded, and that for every leaf $\ell \in \partial\mathbb{U}$, $b_\ell := \lim_{n \rightarrow \infty} \uparrow b_{\ell(n)}$ with $\ell(n)$ the parent of ℓ at generation n .

Theorem 3.10. *The following assertions hold*

(i) *For every $t \geq 0$, there is the identity*

$$\mathcal{E}_x(\mathcal{M}^-(\infty) \mid \bar{\mathcal{F}}_t) = \mathcal{A}(\{\ell \in \partial\mathbb{U} : b_\ell \leq t\}) + M^-(t).$$

(ii) *If $\alpha \geq 0$, the martingale M^- is uniformly integrable under \mathcal{P}_x , and more precisely bounded in L^p for every $p < \omega_+/\omega_-$, and there is the a.s. identity*

$$M^-(\infty) = \mathcal{M}^-(\infty).$$

Further, $\mathcal{A}(\{\ell \in \partial\mathbb{U} : b_\ell < \infty\}) = 0$ a.s.

Proof. (i) We start by observing that, just as in Lemma 3.8,

$$\mathcal{E}_x(\mathcal{M}^-(n) \mid \bar{\mathcal{F}}_t) = \sum_{|u|=n+1} \mathbf{1}_{b_u \leq t} \mathcal{X}_u^{\omega_-}(0) + \sum_{|v| \leq n} \mathbf{1}_{b_v \leq t} \mathcal{X}_v^{\omega_-}(t - b_v).$$

Then, since $\mathcal{M}^-(n)$ converges in $L^1(\mathcal{P}_x)$ to $\mathcal{M}^-(\infty)$ as $n \rightarrow \infty$, the left-hand side converges to $\mathcal{E}_x(\mathcal{M}^-(\infty) \mid \bar{\mathcal{F}}_t)$. In the right-hand-side, $\sum_{|u|=n+1} \mathbf{1}_{b_u \leq t} \mathcal{X}_u^{\omega_-}(0)$ converges to $\mathcal{A}(\{\ell \in \partial\mathbb{U} : b_\ell \leq t\})$ by definition of the intrinsic area measure, and $\sum_{|v| \leq n} \mathbf{1}_{b_v \leq t} \mathcal{X}_v^{\omega_-}(t - b_v)$ to $M^-(t)$.

(ii) When $\alpha \geq 0$, $M^-(t)$ is a martingale and

$$\mathcal{E}_x(M^-(t)) = x^{\omega_-} = \mathcal{E}_x(\mathcal{M}^-(\infty)).$$

By (i), this entails that $\mathcal{A}(\{\ell \in \partial\mathbb{U} : b_\ell \leq t\}) = 0$ a.s. for all $t \geq 0$, and thus the martingale M^- is uniformly integrable. Finally, observe that $\mathcal{M}^-(n)$ is $\bar{\mathcal{F}}_\infty$ -measurable (see the discussion after Lemma 3.2), thus so is $\mathcal{M}^-(\infty)$, and therefore

$$\lim_{t \rightarrow \infty} \mathcal{E}_x(\mathcal{M}^-(\infty) \mid \bar{\mathcal{F}}_t) = \mathcal{M}^-(\infty).$$

Again from (i), this shows the identity $M^-(\infty) = \mathcal{M}^-(\infty)$ a.s. Recall from Lemma 2.4 that then $\mathcal{M}^-(\infty) \in L^p$ if and only if $p < \omega_+/\omega_-$. \square

Remark 3.11. Recall that the law of the martingale \mathcal{M}^- does not depend on the self-similarity parameter α but only on the Laplace exponent Ψ of the Lévy process that drives the evolution of the Eve cell. Actually, its limiting value $\mathcal{M}^-(\infty)$ only depends on the cumulant function κ . Indeed, by Theorem 3.10, we have $\mathcal{M}^-(\infty) = M^-(\infty)$ for $\alpha = 0$, and so $\mathcal{M}^-(\infty)$ only depends on the law of the associated homogeneous growth-fragmentation, which itself is characterized by κ [33].

3.4 Growth-fragmentation equations

We now conclude this section with two results which will not be used in the rest of this work, but might be of independent interest. First, we observe that the moments of the intensity measure μ_t^x can be computed explicitly in the case $\alpha < 0$. Indeed, it follows immediately from Theorem 3.5, the scaling property, and Proposition 1 of [12] that for every $n \geq 1$, we have

$$\int_{(0,\infty)} y^{\omega_+ - n\alpha} \mu_t^x(dy) = x^{\omega_+ - n\alpha} \left(1 + \sum_{\ell=1}^n \frac{\kappa(\omega_+ - \alpha n) \cdots \kappa(\omega_+ - \alpha(n - \ell + 1))}{\ell!} (tx^\alpha)^\ell \right). \quad (17)$$

In the spectrally negative case when cell processes have no positive jumps, that is, equivalently when $\Lambda((0, \infty)) = 0$, this determines μ_t^x .

We finally point out that in the spectrally negative case, Theorem 3.5 establishes a connection between growth-fragmentation processes and growth-fragmentation equations (see [10] and references therein).

Corollary 3.12. *Suppose that the killing rate is $\mathbf{k} = 0$ and that $\Lambda((0, \infty)) = 0$. For every \mathcal{C}^∞ function $f : (0, \infty) \rightarrow \mathbb{R}$ with compact support and $t \geq 0$, there is the identity*

$$\langle \mu_t^x, f \rangle = f(x) + \int_0^t \langle \mu_s^x, \mathcal{L}f \rangle ds,$$

where $\mathcal{L}f(x) = x^\alpha \mathcal{L}_0 f(x)$ and $\mathcal{L}_0 f(x)$ is given by

$$\frac{1}{2} \sigma^2 x^2 f''(x) + \left(b + \frac{1}{2} \sigma^2 \right) x f'(x) + \int_{(-\infty, 0)} (f(e^y x) + f((1 - e^y)x) - f(x) + x f'(x)(1 - e^y)) \Lambda(dy).$$

Proof. This follows from Corollary 4.1 in [10] when $\alpha < 0$, and from Corollary 4.8 there when $\alpha > 0$. In the homogeneous case $\alpha = 0$, one can use Theorem 3.1 in [10] combined with an application of Esscher's transform. We stress that the function κ given here in (5) has a slightly different form in (1.6) in [10], which induces that, in the present statement, the coefficient for the first derivative f' is $(b + \frac{1}{2} \sigma^2)$ and not simply b as one might have expected. \square

4 Spinal decompositions

Following Lyons, Pemantle & Peres [30], the main purpose of this section will be to describe the growth-fragmentation process under the tilted probability measure which is associated with the intrinsic martingales \mathcal{M}^+ and \mathcal{M}^- . Recall that Lyons, Pemantle & Peres used additive martingales in branching random walks to tag a leaf at random in the genealogical tree of the process. The ancestral lineage of the tagged leaf forms the so-called spine, which is viewed as a branch consisting of tagged particles. Roughly speaking, the spinal decomposition claims that the untagged particles evolve as ordinary particles in the branching random walk, whereas tagged particles reproduce according to a biased reproduction law, and the tagged child at the next generation is then picked by biased sample from the children of the tagged parent.

Informally, using \mathcal{M}^+ to perform the probability tilting can be thought of as conditioning a distinguished cell to grow indefinitely, whereas the Malthusian martingale \mathcal{M}^- rather corresponds to tagging a cell randomly according to the intrinsic area measure \mathcal{A} . As the arguments are very similar, we shall provide complete proofs in the first case, and skip details in the second.

4.1 Conditioning on indefinite growth

In the setting of cell systems, we first define as follows a probability measure $\widehat{\mathcal{P}}_x^+$ describing the joint distribution of a cell system $\mathcal{X} = (\mathcal{X}_u : u \in \mathbb{U})$ (recall that we use canonical notation) and a leaf $\mathcal{L} \in \partial\mathbb{U}$, where $\partial\mathbb{U}$ denotes the boundary of the Ulam tree, that is the space $\mathbb{N}^{\mathbb{N}}$ of infinite sequences of positive integers. Recall also that \mathcal{G}_n denotes the sigma-field generated by the cells with generation at most n . To start with, for every $n \geq 0$, the law of $(\mathcal{X}_u : |u| \leq n)$ under $\widehat{\mathcal{P}}_x^+$ is absolutely continuous with respect to the restriction of \mathcal{P}_x to \mathcal{G}_n , with density $x^{-\omega_+} \mathcal{M}^+(n)$, viz.

$$\widehat{\mathcal{P}}_x^+(\Gamma_n) = x^{-\omega_+} \mathcal{E}_x(\mathcal{M}^+(n) \mathbb{1}_{\Gamma_n}) \quad \text{for every event } \Gamma_n \in \mathcal{G}_n.$$

We then discuss the tagged leaf \mathcal{L} . First, for every leaf $\ell \in \partial\mathbb{U}$, we write $\ell(n)$ for the parent of ℓ at generation n , that is the sequence of the n first elements of ℓ . Then the conditional law of the parent $\mathcal{L}(n+1)$ of the tagged leaf at generation $n+1$ is given by

$$\widehat{\mathcal{P}}_x^+(\mathcal{L}(n+1) = v \mid \mathcal{G}_n) = \frac{\mathcal{X}_v^{\omega_+}(0)}{\mathcal{M}^+(n)}, \quad \text{for every } v \text{ at generation } |v| = n+1. \quad (18)$$

The coherence of this definition, that is the fact that if the conditional distribution of $\mathcal{L}(n+1)$ given \mathcal{G}_n fulfills (18), then its mother $\mathcal{L}(n)$ fulfills the analog identity, is ensured by the martingale property of \mathcal{M}^+ and the branching structure of cell systems. Recalling that \mathcal{M}^+ can be viewed as an additive martingale for the branching random walk \mathcal{Z}_n which is given by the logarithm of the sizes at births of cells at generation n , we see that the sequence $(-\ln \mathcal{L}(n) : n \geq 0)$ corresponds precisely to the spine in the framework considered by Lyons, Pemantle & Peres [30]. Roughly speaking, this provides the discrete (more precisely, generational) skeleton of the tagged cell, which we now introduce.

The birth-times $b_{\ell(n)}$ of the cells on the ancestral lineage of a leaf $\ell \in \partial\mathbb{U}$ form an increasing sequence, which converges to $b_\ell := \lim_{n \rightarrow \infty} b_{\ell(n)}$. Focussing on the tagged leaf \mathcal{L} , we set $\hat{\mathcal{X}}(t) = \partial$ (recall that ∂ stands for a cemetery point) for $t \geq b_\mathcal{L}$ and

$$\hat{\mathcal{X}}(t) = \mathcal{X}_{\mathcal{L}(n_t)}(t - b_{\mathcal{L}(n_t)}) \quad \text{for } t < b_\mathcal{L},$$

where n_t denotes the generation of the parent of the tagged leaf at time t , that is the unique integer $n \geq 0$ such that $b_{\mathcal{L}(n)} \leq t < b_{\mathcal{L}(n+1)}$. We should think of $\hat{\mathcal{X}}(t)$ as the size of the *tagged cell* at time t ; understanding its evolution provides the key to many properties of the law $\widehat{\mathcal{P}}_x^+$.

We first point out that arguments close to those developed in the proof of Lemma 3.8 show that the conditional distribution of $\hat{\mathcal{X}}(t)$ given $\bar{\mathcal{F}}_t$ has a simple expression as a power-size biased sample from $\mathbf{X}(t)$. Observe from the very definition (18), that for every \mathcal{G}_n -measurable random variable $\mathcal{B}(n) \geq 0$, there is the identity

$$x^{\omega_+} \hat{\mathcal{E}}_x^+(f(\mathcal{X}_{\mathcal{L}_{n+1}}(0))\mathcal{B}(n)) = \mathcal{E}_x \left(\sum_{|u|=n+1} \mathcal{X}_u^{\omega_+}(0) f(\mathcal{X}_u(0)) \mathcal{B}(n) \right).$$

We obtain an analog identity when generations are replaced by times.

Proposition 4.1. *For every $t \geq 0$, every measurable function $f : [0, \infty) \rightarrow [0, \infty)$, and every $\bar{\mathcal{F}}_t$ -measurable random variable $B(t) \geq 0$, we have*

$$x^{\omega_+} \hat{\mathcal{E}}_x^+(f(\hat{\mathcal{X}}(t))B(t)) = \mathcal{E}_x \left(\sum_{i=1}^{\infty} X_i^{\omega_+}(t) f(X_i(t)) B(t) \right),$$

with the usual convention that $f(\partial) = 0$.

Proof. Let us first assume that $B(t)$ is $\bar{\mathcal{F}}_t \wedge \mathcal{G}_k$ -measurable for some fixed $k \in \mathbb{N}$. Since $f(\partial) = 0$ and $\hat{\mathcal{X}}(t) = \partial$ for $t > b_{\mathcal{L}}$, we have

$$\hat{\mathcal{E}}_x^+(f(\hat{\mathcal{X}}(t))B(t)) = \lim_{n \rightarrow \infty} \hat{\mathcal{E}}_x^+(f(\hat{\mathcal{X}}(t))B(t) \mathbb{1}_{b_{\mathcal{L}(n+1)} > t}).$$

Then, provided that $n > k$, we have from the definition of the tagged leaf that

$$\hat{\mathcal{E}}_x^+(f(\hat{\mathcal{X}}(t))B(t) \mathbb{1}_{b_{\mathcal{L}(n+1)} > t}) = x^{-\omega_+} \mathcal{E}_x \left(\sum_{|u|=n+1} \mathcal{X}_u^{\omega_+}(0) \mathbb{1}_{b_u > t} f(\mathcal{X}_{u(t)}(t - b_{u(t)})) B(t) \right),$$

where, for every cell u with $b_u > t$, $u(t)$ denotes the most recent ancestor of u which is alive at time t . Just as in the proof of Lemma 3.8, we decompose the family of cells u at generation $n+1$ which are born after time t into sub-families having the same most recent ancestor alive at time t . Applications of the branching property stated in Lemma 3.2 and of the martingale property of \mathcal{M}^+ , we get

$$\mathcal{E}_x \left(\sum_{|u|=n+1} \mathcal{X}_u^{\omega_+}(0) \mathbb{1}_{b_u > t} f(\mathcal{X}_{u(t)}(t - b_{u(t)})) B(t) \right) = \mathcal{E}_x \left(\sum_{|v| \leq n} \mathbb{1}_{b_v \leq t} \mathcal{X}_v^{\omega_+}(t - b_v) f(\mathcal{X}_v(t - b_v)) B(t) \right).$$

Letting $n \rightarrow \infty$ and applying monotone convergence establish the formula of the statement when $B(t)$ is $\bar{\mathcal{F}}_t \wedge \mathcal{G}_k$ -measurable. The case when we only assume that $B(t)$ is $\bar{\mathcal{F}}_t$ -measurable then follows by a monotone class argument. \square

We next turn our attention to the so-called *spinal decomposition* popularized by Lyons, Pemantle & Peres for branching random walks. In words, we follow the tagged cell as time passes, and for each of its negative jumps, we record the entire growth-fragmentation process which that jump generates (that is, as usual, we interpret a negative jump as a birth event, and then record the growth-fragmentation process corresponding to the new-born cell). Roughly speaking, we shall see in the next theorem, that under $\hat{\mathcal{P}}_1^+$, the law of the tagged cell is given by that of a certain self-similar Markov process, and that conditionally on the path of the tagged cell, the growth-fragmentation processes generated by the negative jumps of $\hat{\mathcal{X}}$ are independent with the law \mathbb{P}_x , where x is the (absolute) size of the jump.

In order to give a precise statement, we need to label the negative jumps of $\hat{\mathcal{X}}$. If those jumps times were isolated, then we could simply enumerate them in increasing order. Alternatively, if the tagged cell

converged to 0, then we could enumerate its negative jumps in the increasing order of their absolute sizes. However this is not the case in general, and we shall therefore introduce a deterministic algorithm, which is tailored for our purpose.

We first label each negative jump of the tagged cell by a pair (n, j) , where $n \geq 0$ denotes the generation of the tagged cell immediately before the jump occurs, and $j \geq 1$ the rank of that jump amongst the negative jumps which occurred while the generation of the tagged cell equals n , including the terminal jump when the generation of the tagged cell increases by one unit. We then write $\hat{\mathbf{X}}_{n,j} = (\hat{\mathbf{X}}_{n,j}(t) : t \geq 0)$ for the growth-fragmentation generated by the (n, j) -jump. Specifically, on the one hand, if the generation of the tagged cell does not increase during the (n, j) -jump, and if u denotes the label of the cell born at this birth event, then

$$\hat{\mathbf{X}}_{n,j}(t) = \{\{\mathcal{X}_{uv}(t + b_u - b_{uv}) : v \in \mathbb{U}, b_{uv} \leq t + b_u < b_{uv} + \zeta_{uv}\}\}.$$

On the other hand, if the (n, j) -jump occurs at an instant, say \mathcal{T} , at which the generation of the tagged cell increases, and if u is the label of the tagged cell immediately before the jump, then $\hat{\mathbf{X}}_{n,j}$ is the growth-fragmentation process generated by the cell u after time \mathcal{T} , i.e.

$$\hat{\mathbf{X}}_{n,j}(t) = \{\{\mathcal{X}_{uv}(t + \mathcal{T} - b_{uv}) : v \in \mathbb{U}, b_{uv} \leq t + \mathcal{T} < b_{uv} + \zeta_{uv}\}\}.$$

For definitiveness, we agree as usual that $\hat{\mathbf{X}}_{n,j}(t) \equiv \partial$ when the (n, j) -jump does not exist. It should be plain that the entire growth-fragmentation process \mathbf{X} can be recovered from the process $(\hat{\mathcal{X}}(t), n_t)_{0 \leq t < b_{\mathcal{L}}}$ of the tagged cell and its generation, and the family of processes $(\hat{\mathbf{X}}_{n,j} : n \geq 0, j \geq 1)$.

Next, recall Lemma 3.4 and the notation there; in particular η^+ denotes a Lévy process with Lévy measure Π . By the Lévy-Itô decomposition, we can think of the negative jumps of η^+ as resulting from the superposition of two independent Poisson point processes on $\mathbb{R}_+ \times (-\infty, 0)$, the first with intensity $e^{x\omega_+} dt \Lambda(dx)$, and the second with intensity $e^{x\omega_+} dt \tilde{\Lambda}(dx)$. Because $\kappa(\omega_+) = 0$, (5) entails that

$$\int_{(-\infty, 0)} e^{x\omega_+} \tilde{\Lambda}(dx) = \int_{(-\infty, 0)} (1 - e^x)^{\omega_+} \Lambda(dx) = -\Psi(\omega_+) < \infty,$$

so if we mark the negative jumps of η^+ that correspond to the second point process, then the number of marked jumps up to time $t \geq 0$, $N^+(t)$, forms a Poisson process with intensity $-\Psi(\omega_+)$. Recall also that we defined Y^+ just after Lemma 3.4 as the self-similar Markov process with characteristics (Φ^+, α) started from 1. Specifically we write the Lamperti time-substitution (recall (3))

$$\tau_t^+ := \inf \left\{ r \geq 0 : \int_0^r \exp(-\alpha \eta^+(s)) ds \geq t \right\} \quad \text{for } 0 \leq t < I^+ := \int_0^\infty \exp(-\alpha \eta^+(s)) ds,$$

and

$$Y^+(t) := \exp(\eta^+(\tau_t^+))$$

with the usual convention that $Y^+(t) = \partial$ for $t \geq I^+$. We are now able to claim:

Theorem 4.2. *The distribution of $(\hat{\mathcal{X}}(t), n_t)_{0 \leq t < b_{\mathcal{L}}}$ under $\hat{\mathcal{P}}_1^+$ is the same as that of*

$$(Y^+(t), N^+(\tau_t^+))_{0 \leq t < I^+}.$$

Further, conditionally on $(\hat{\mathcal{X}}(t), n_t)_{0 \leq t < b_{\mathcal{L}}}$, the processes $\hat{\mathbf{X}}_{n,j}$ for $n \geq 0$ and $j \geq 1$ are independent, and more precisely, each $\hat{\mathbf{X}}_{n,j}$ has the (conditional) law \mathbb{P}_x , with x the absolute size of the jump of $\hat{\mathcal{X}}$ with label (n, j) .

Proof. We need only to establish the statements for $\alpha = 0$, as the general case then follows from Lamperti's transformation. We focus first on the first claim and observe, by the branching property of cell systems and the Markov property of Poisson point processes, that it suffices to verify that the distribution of $(\hat{\mathcal{X}}(t))_{0 \leq t \leq b_{\mathcal{L}(1)}}$ under $\hat{\mathcal{P}}_1^+$ is the same as that of $(\exp(\eta^+(t)))_{0 \leq t \leq T_1}$, where T_1 denotes the instant when the first atom of the Poisson point process with intensity $e^{x\omega_+} dt \tilde{\Lambda}(dx)$ arises. We stress that the terminal values $b_{\mathcal{L}(1)}$ and T_1 are included in the life-interval of those processes.

In this direction, we use Lemma 3.4(i) and decompose the Lévy process η^+ as the sum of two independent Lévy processes,

$$\eta^+(t) = \xi^+(t) + \nu(t),$$

where $(\nu(t))_{t \geq 0}$ is a compound Poisson process with Lévy measure $e^{x\omega_+} \tilde{\Lambda}(dx)$. That is, we have for every $q, t \geq 0$

$$\begin{aligned} E(\exp(q\nu(t))) &= \exp\left(t \int_{(-\infty, 0)} (e^{qx} - 1) e^{x\omega_+} \tilde{\Lambda}(dx)\right) \\ &= \exp\left(t \int_{(-\infty, 0)} ((1 - e^x)^{q+\omega_+} - (1 - e^x)^{\omega_+}) \Lambda(dx)\right). \end{aligned}$$

The identity

$$E(\exp(q\xi^+(t))) = E(\exp(q\eta^+(t))) / E(\exp(q\nu(t)))$$

combined with (5) then entails that the Laplace exponent of the Lévy process ξ^+ is $\Psi(q + \omega_+) - \Psi(\omega_+)$.

Next observe that in this setting, $T_1 = \inf\{t \geq 0 : \Delta\cdot\nu(t) < 0\}$. Elementary properties of Poisson random measures then show that $\Delta\cdot\nu(T_1) = \Delta\cdot\eta^+(T_1)$ has the law $-\Psi(\omega_+)^{-1} e^{x\omega_+} \tilde{\Lambda}(dx)$, and is independent of the process $(\eta^+(t))_{0 \leq t < T_1} = (\xi^+(t))_{0 \leq t < T_1}$ (note that the right-extremity T_1 of the time interval is now excluded). The latter has the distribution of the Lévy process ξ^+ killed at an independent exponential time with parameter $-\Psi(\omega_+)$, and we deduce from above that its Laplace exponent is $\Psi^+(q) := \Psi(q + \omega_+)$. This entirely describes the law of $(\eta^+(t))_{0 \leq t \leq T_1}$, and we shall now check that the law of $(\ln \hat{\mathcal{X}}(t))_{0 \leq t \leq b_{\mathcal{L}(1)}}$ under $\hat{\mathcal{P}}_1^+$ can be depicted in the same way.

Consider an arbitrary functional $F \geq 0$ on the space of finite càdlàg paths, and $g : (-\infty, 0) \rightarrow \mathbb{R}_+$ measurable. We aim at computing the quantity

$$\hat{\mathcal{E}}_1^+ (F(\ln \mathcal{X}_{\mathcal{L}}(s) : 0 \leq s < b_{\mathcal{L}(1)}) g(\ln \mathcal{X}_{\mathcal{L}(1)}(0) - \ln \mathcal{X}_{\mathcal{L}}(\mathcal{L}(1)-))),$$

which, by the definition of the tagged leaf, we can express in the form (recall that we place ourselves in the homogeneous case $\alpha = 0$)

$$\begin{aligned} &\mathcal{E}_1 \left(\sum_{t>0} F(\ln \mathcal{X}_{\mathcal{L}}(s) : 0 \leq s < t) g(\ln |\Delta\cdot\mathcal{X}_{\mathcal{L}}(t)| - \ln \mathcal{X}_{\mathcal{L}}(t-)) |\Delta\cdot\mathcal{X}_{\mathcal{L}}(t)|^{\omega_+} \right) \\ &= E \left(\sum_{t>0} F(\xi(s) : 0 \leq s < t) g(\ln(1 - \exp(\Delta\cdot\xi(t)))) |1 - \exp(\Delta\cdot\xi(t))|^{\omega_+} \exp(\omega_+ \xi(t-)) \right). \end{aligned}$$

We can now compute this expression using the Lévy-Itô decomposition of ξ and the compensation formula. We get

$$\begin{aligned} &E \left(\int_0^\infty F(\xi(s) : 0 \leq s < t) \exp(\xi(t)\omega_+) dt \right) \int_{(-\infty, 0)} g(\ln(1 - e^x)) |1 - e^x|^{\omega_+} \Lambda(dx) \\ &= E \left(\int_0^\infty F(\xi(s) : 0 \leq s < t) \exp(\xi(t)\omega_+) dt \right) \int_{(-\infty, 0)} g(y) e^{y\omega_+} \tilde{\Lambda}(dy). \end{aligned}$$

This shows that under $\hat{\mathcal{P}}_1^+$, the variable $\ln \mathcal{X}_{\mathcal{L}(1)}(0) - \ln \mathcal{X}_{\emptyset}(\mathcal{L}(1)-)$ and the process $(\ln \hat{\mathcal{X}}(t))_{0 \leq t < b_{\mathcal{L}(1)}}$ are independent. The former has the law $-\Psi(\omega_+)^{-1} e^{x\omega_+} \hat{\Lambda}(dx)$, and the latter that of ξ killed according to the multiplicative functional $\exp(\xi(t)\omega_+)$. It is immediately seen (and well-known) that this yields a Lévy process with Laplace exponent $\Psi^+(q) = \Psi(q + \omega_+)$, and this completes the check of the first assertion.

For the second assertion about the conditional distribution of the families of growth-fragmentations which stem from the spine, we shall only check that under $\hat{\mathcal{P}}_1^+$, conditionally on $(\hat{\mathcal{X}}(t))_{0 \leq t \leq b_{\mathcal{L}(1)}}$, the processes $\hat{\mathbf{X}}_{0,j}$ for $j \geq 1$ are independent, and more precisely, each $\hat{\mathbf{X}}_{0,j}$ has the (conditional) law \mathbb{P}_x , with x the absolute size of the j -th largest negative jump of $\hat{\mathcal{X}}$ on $[0, b_{\mathcal{L}(1)}]$. Indeed, the more general assertion in the statement then follows easily from the branching property of cell systems.

Consider an arbitrary functional $F \geq 0$ on the space of finite càdlàg paths, and for every $j \geq 1$, a functional $G_j \geq 0$ of the space of multi-set valued paths, and for every $x > 0$, set

$$g_j(x) := \mathbb{E}_x(G_j(\mathbf{X})).$$

We aim at checking that there is the identity

$$\hat{\mathcal{E}}_1^+ \left(F(\mathcal{X}_{\emptyset}(s) : 0 \leq s \leq b_{\mathcal{L}(1)}) \prod_{j=1}^{\infty} G_j(\hat{\mathbf{X}}_{0,j}) \right) = \hat{\mathcal{E}}_1^+ \left(F(\mathcal{X}_{\emptyset}(s) : 0 \leq s \leq b_{\mathcal{L}(1)}) \prod_{j=1}^{\infty} g_j(D_j(t)) \right),$$

where $(D_j(t))_{j \geq 1}$ denotes the sequence formed by the absolute values of the negative jumps of \mathcal{X}_{\emptyset} that occurred strictly before time t and the value of \mathcal{X}_{\emptyset} at time t , ranked in the non-increasing order.

By the definition of the tagged cell, the left hand side equals

$$\mathcal{E}_1 \left(\sum_{t>0} F(\mathcal{X}_{\emptyset}(s) : 0 \leq s \leq t) |\Delta_- \mathcal{X}_{\emptyset}(t)|^{\omega_+} \prod_{j=1}^{\infty} G_j(\hat{\mathbf{X}}_{0,j}) \right),$$

and then, by the branching property of cell systems under \mathcal{P}_1 and the definition of $g_j(x)$, the latter quantity can be expressed as

$$\mathcal{E}_1 \left(\sum_{t>0} F(\mathcal{X}_{\emptyset}(s) : 0 \leq s \leq t) |\Delta_- \mathcal{X}_{\emptyset}(t)|^{\omega_+} \prod_{j=1}^{\infty} g_j(D_j(t)) \right).$$

Again, by the definition of the tagged cell, we now see that this quantity coincides with

$$\hat{\mathcal{E}}_1^+ \left(F(\mathcal{X}_{\emptyset}(s) : 0 \leq s \leq b_{\mathcal{L}(1)}) \prod_{j=1}^{\infty} g_j(D_j(t)) \right),$$

and this is precisely what we wanted to show. \square

Theorem 4.2 states in particular that under $\hat{\mathcal{P}}_1^+$, the tagged cell has the distribution of the self-similar Markov process Y^+ with characteristics (Φ^+, α) , and this entails that the lifetime $b_{\mathcal{L}}$ of the tagged cell is infinite $\hat{\mathcal{P}}_1^+$ -a.s. when $\alpha \leq 0$, whereas $b_{\mathcal{L}} < \infty$ for $\alpha > 0$. In both cases $\lim_{t \rightarrow b_{\mathcal{L}}} \hat{\mathcal{X}}(t) = \infty$, $\hat{\mathcal{P}}_1^+$ -a.s.

4.2 Starting the growth-fragmentation with indefinite growth from 0

In this section, we shall always assume that $\alpha \leq 0$, so the tagged cell does not explode, and we know further from Corollary 3.7 that M^+ is a \mathbb{P}_x -martingale. We write \mathbb{P}_x^+ for the law of the growth-fragmentation \mathbf{X} under $\hat{\mathcal{P}}_x^+$, and note from Lemma 3.8 that there is the relation of absolute continuity

$$x^{\omega_+} \mathbb{E}_x^+(A(t)) = \mathbb{E}_x(M^+(t)A(t))$$

for every \mathcal{F}_t -measurable variable $A(t) \geq 0$. This enables us in particular to view $\mathbf{X}(t)$ under \mathbb{P}_x^+ as a random non-increasing null sequence. We shall now use the spinal decomposition to investigate the asymptotic behavior of \mathbb{P}_x^+ as $x \rightarrow 0+$

Roughly speaking, the spinal decomposition consists in assigning the role of Eve to the tagged cell $\hat{\mathcal{X}}$ rather than to \mathcal{X}_\emptyset in the description of cell systems. Theorem 4.2 thus incites us to introduce another distribution for cell-systems, denoted by \mathcal{Q}_x^+ , which is defined as follows. Under \mathcal{Q}_x^+ , the Eve cell \mathcal{X}_\emptyset follows the law Q_x^+ of the self-similar Markov process Y^+ with characteristics (Φ^+, α) . Recall that $\lim_{t \rightarrow \infty} Y^+(t) = \infty$ a.s., so the absolute values of the negative jumps of Y^+ cannot be ranked in the non increasing order. However, this only a minor issue; indeed, we may use for instance the easy fact that $\lim_{t \rightarrow \infty} e^{-t} Y^+(t) = 0$ a.s., and then rank the jumps sizes and times of $t \mapsto -e^{-t} \mathcal{X}_\emptyset(t)$ in the non-increasing order of their sizes, say $(e^{-\beta_1} y_1, \beta_1), (e^{-\beta_2} y_2, \beta_2), \dots$. We get the initial sizes and birth-times of the daughter cells at the first generation. That is, conditionally on \mathcal{X}_\emptyset , the processes $\mathcal{X}_1, \mathcal{X}_2, \dots$ are independent processes with laws P_{y_1}, P_{y_2}, \dots . We iterate for the next generations just as in Sec. 2.2. We stress that only the mother cell evolves according to the self-similar Markov process with characteristics (Φ^+, α) , whereas the daughters, granddaughters, ... all evolve according to the self-similar Markov process with characteristics (Ψ, α) . Theorem 4.2 entails in particular that the law \mathbb{P}_x^+ of the growth-fragmentation \mathbf{X} is the same under \mathcal{Q}_x^+ as under $\hat{\mathcal{P}}_x^+$ (genealogies are of course different; in other words the cell processes have different laws under \mathcal{Q}_x^+ as under $\hat{\mathcal{P}}_x^+$, even though the growth-fragmentations they induce have the same distribution. This feature has been analyzed in depth by Shi [33]).

An important motivation for introducing the law \mathcal{Q}_x^+ for cell systems, is that the law Q_x^+ of the Eve cell \mathcal{X}_\emptyset possesses a non-degenerate weak limit Q_0^+ as $x \rightarrow 0+$. More precisely, since $\Phi^+(0) = 0$, $(\Phi^+)'(0) > 0$ and $\alpha < 0$, the family of laws of Markov processes $(Q_x^+)_{x \geq 0}$ fulfills the Feller property, see e.g. [11]. It follows readily that as $x \rightarrow 0+$, \mathcal{Q}_x^+ converges weakly, in the sense of finite-dimensional distributions for families indexed by \mathbb{U} , towards \mathcal{Q}_0^+ , the law of the cell system under which the Eve cell \mathcal{X}_\emptyset has the law Q_0^+ and all the other cells are self-similar Markov processes with characteristics (Ψ, α) .

We shall now show that under \mathcal{Q}_0^+ , the multisets $\mathbf{X}(t)$ can be still viewed as non-increasing null sequences, a.s. Recall that for every $q > 0$ and every multiset $\mathbf{x} = \{\{x_i : i \in \mathbb{N}\}\}$, we write $F_q(\mathbf{x}) := \sum_{i \in \mathbb{N}} x_i^q$ for the sum of the elements of \mathbf{x} raised to the power q and repeated according to their multiplicity.

Lemma 4.3. *Assume $\alpha < 0$. For every $t \geq 0$ and $q > 0$ with $\kappa(q) \leq 0$, we have $F_q(\mathbf{X}(t)) < \infty$, \mathcal{Q}_0^+ -a.s.*

Proof. Consider the (multi-)set formed by the cells alive at time t once the Eve cell has been removed, $\mathbf{X}^*(t) := \mathbf{X}(t) \setminus \{\mathcal{X}_\emptyset(t)\}$. Since we know from Corollary 3.7 that $F_q(\mathbf{X}(t))$ is a \mathbb{P}_x -supermartingale, it follows from the construction of \mathcal{Q}_0^+ that the conditional expectation under \mathcal{Q}_0^+ of $F_q(\mathbf{X}^*(t))$ given the Eve cell is bounded from above by

$$\mathcal{Q}_0^+ (F_q(\mathbf{X}^*(t)) \mid \mathcal{X}_\emptyset) \leq \sum_{0 < s < t} |\Delta_- \mathcal{X}_\emptyset(s)|^q.$$

To complete the proof, we just need to check that $\sum_{0 < s < t} |\Delta_- Y^+(s)|^q < \infty$, Q_0^+ -a.s., which is easy from Lamperti's transformation.

Specifically, recall first that the Lévy measure of Φ^+ is given in Lemma 3.4(i), and note from the fact that $\kappa(q) < \infty$ that

$$c^+(q) := \int_{(-\infty, 0)} (1 - e^x)^q e^{x\omega_+} (\Lambda(dx) + \tilde{\Lambda}(dx)) < \infty.$$

Next, recall that η^+ is the Lévy process associated with Y^+ by Lamperti's transformation. An application

of the Lévy-Itô decomposition of η^+ shows that the predictable compensator of the increasing process

$$t \mapsto \sum_{0 < s \leq t} \exp(q\eta^+(s-))(1 - \exp(\Delta\eta^+(s)))^q$$

is given by

$$t \mapsto c^+(q) \int_0^t \exp(q\eta^+(s)) ds.$$

When we combine this with Lamperti's transformation, this entails that the predictable compensator under \mathcal{Q}_0^+ of $t \mapsto \sum_{0 < s \leq t} |\Delta Y^+(s)|^q < \infty$ is $t \mapsto c^+(q) \int_0^t (Y^+(s))^{q+\alpha} ds$. That the latter is indeed a well-defined (i.e. with finite values) process for every $q > 0$ is easy and can be seen e.g. from the results of Chaumont and Pardo [19]. This shows that $F_q(\mathbf{X}^*(t)) < \infty$ \mathcal{Q}_0^+ -a.s., and the lemma is proved. \square

Lemma 4.3 enables us to rank the elements of $\mathbf{X}(t)$ (repeated according to their multiplicities) in the non-increasing order, and we then obtain a null sequence, \mathcal{Q}_0^+ -a.s. We write \mathbb{P}_0^+ for the law of \mathbf{X} under \mathcal{Q}_0^+ , and claim the following limit theorem in the spectrally negative case (we believe that this restriction should be essentially superfluous, however it makes the argument somewhat simpler).

Corollary 4.4. *Assume $\alpha < 0$ and that cells have no positive jumps a.s., that is $\Lambda((0, \infty)) = 0$. The family of probability measures \mathbb{P}_x^+ converges weakly towards \mathbb{P}_0^+ as $x \rightarrow 0$, in the sense of finite dimensional distributions for processes with values in the space of non-increasing null sequences.*

Proof. We work under \mathcal{Q}_0^+ . Introduce the first passage times $T_x := \inf\{t \geq 0 : \mathcal{X}_\emptyset(t) = x\}$ for every $x > 0$ and observe that $T_x < \infty$ and $\lim_{x \rightarrow 0} T_x = 0$ a.s. (because these properties hold for the self-similar Markov process Y^+ under \mathcal{Q}_0^+ ; note that we use here the assumption of absence of positive jumps). Imagine that we kill all the daughter cells \mathcal{X}_i for $i \in \mathbb{N}$ which are born before T_x together with their descent, so that the killed cells are those labelled by

$$\mathbb{U}_x := \bigcup_{i=1}^{\infty} \{v = iu : b_i \leq T_x \text{ \& } u \in \mathbb{U}\}.$$

First, if we define

$$\mathbf{X}^{(x)}(t) := \{\mathcal{X}_u(t - b_u) : u \in \mathbb{U} \setminus \mathbb{U}_x \text{ \& } t \geq b_u\},$$

i.e. $\mathbf{X}^{(x)}(t)$ is the family of surviving cells which are alive at time t , then by the Markov property of the Eve cell, we see that the shifted process $(\mathbf{X}^{(x)}(t + T_x))_{t \geq 0}$ has the law \mathbb{P}_x^+ . Second, for every $q > 0$ with $\kappa(q) < 0$, the same argument as in the proof of Lemma 4.3 yields the bound

$$\mathcal{Q}_0^+ \left(F_q(\mathbf{X}(t) \setminus \mathbf{X}^{(x)}(t)) \mid \mathcal{X}_\emptyset \right) \leq \sum_{0 < s < T_x \wedge t} |\Delta \mathcal{X}_\emptyset(s)|^q,$$

from which we infer

$$\lim_{x \rightarrow 0+} F_q(\mathbf{X}(t) \setminus \mathbf{X}^{(x)}(t)) = \lim_{x \rightarrow 0+} \sum_{u \in \mathbb{U}^{(x)}} \mathbb{1}_{t \geq b_u} \mathcal{X}_u^q(t - b_u) = 0 \quad \text{a.s.}$$

Third, we recall from Lemma 4.3 that the family $\mathbf{X}(t)$ is q -summable a.s. The branching property enables us to apply the same argument as in the proof of Corollary 4 in [5], and we assert that

$$\lim_{x \rightarrow 0+} \sum_{u \in \mathbb{U}} |\mathbb{1}_{t \geq b_u} \mathcal{X}_u(t - b_u) - \mathbb{1}_{t+T_x \geq b_u} \mathcal{X}_u(t + T_x - b_u)|^q = 0 \quad \text{in probability.}$$

Combining this with the second observation above, we deduce that

$$\lim_{x \rightarrow 0+} \sum_{u \in \mathbb{U}} |\mathbb{1}_{t \geq b_u} \mathcal{X}_u(t - b_u) - \mathbb{1}_{u \notin \mathbb{U}_x} \mathbb{1}_{t+T_x \geq b_u} \mathcal{X}_u(t + T_x - b_u)|^q = 0 \quad \text{in probability.}$$

Our claim now follows easily from the fact that the family

$$\{\{\mathcal{X}_u(t + T_x - b_u) : u \in \mathbb{U} \setminus \mathbb{U}_x \text{ \& } t + T_x \geq b_u\}\}$$

has the same law as $\mathbf{X}(t)$ under \mathbb{P}_x^+ (this is the first observation above). \square

Here is an application of the preceding result to the extinction time. Specifically, recall that when $\alpha < 0$, the growth-fragmentation \mathbf{X} under \mathbb{P}_x for $x > 0$ is absorbed at $(0, 0 \dots)$ after an a.s. finite time $\epsilon := \inf\{t \geq 0 : \mathbf{X}(t) \equiv 0\}$; see Corollary 3 in [5]. We now obtain the following polynomial lower-bound estimates for the tail distribution of this absorption time, which contrasts sharply with the exponential decay proved by Haas for self-similar (pure) fragmentation processes (see Proposition 14 in [23]). We use the notation $f \asymp g$ for a pair of functions $f, g : \mathbb{R}_+ \rightarrow (0, \infty)$ such that the ratio $f(t)/g(t)$ remains bounded away from 0 and ∞ as $t \rightarrow \infty$. For the sake of simplicity, we concentrate on the spectrally negative case, although this restriction is probably superfluous.

Corollary 4.5. *Assume $\alpha < 0$ and that cells have no positive jumps a.s., that is $\Lambda((0, \infty)) = 0$. Then we have $\mathbb{P}_1(\epsilon > t) \asymp t^{\omega_+/\alpha}$.*

Proof. We shall first prove the following lower bound

$$\liminf_{t \rightarrow \infty} t^{-\omega_+/\alpha} \mathbb{P}_1(\epsilon > t) \geq \mathbb{E}_0^+(1/M^+(1)) > 0.$$

We use self-similarity and write $\mathbb{P}_1(\epsilon > t) = \mathbb{P}_{t^{1/\alpha}}(\epsilon > 1)$. Then recalling that extinction does not occur $\widehat{\mathcal{P}}_x^+$ -a.s., we have from Proposition 4.1 that

$$t^{-\omega_+/\alpha} \mathbb{P}_1(\epsilon > t) = \widehat{\mathcal{E}}_{t^{1/\alpha}}^+(1/M^+(1)) = \mathbb{E}_{t^{1/\alpha}}^+(1/M^+(1)).$$

We then point out that the calculations in the proof of Corollary 4.4 show that the law of $M^+(t)$ under \mathbb{P}_x^+ converges to that of $M^+(t)$ under \mathbb{P}_0^+ as $x \rightarrow 0+$, and that because $\alpha < 0$, $t^{1/\alpha} \rightarrow 0+$ as $t \rightarrow \infty$. Then, for any $a > 0$, we have

$$\lim_{t \rightarrow \infty} \mathbb{E}_{t^{1/\alpha}}^+(a \wedge (1/M^+(1))) = \mathbb{E}_0^+(a \wedge (1/M^+(1)))$$

and the right-hand side is a strictly positive quantity since we know from Lemma 4.3 that $M^+(1) < \infty$ \mathbb{P}_0^+ -a.s.

In order to establish the upper-bound

$$\limsup_{t \rightarrow \infty} t^{-\omega_+/\alpha} \mathbb{P}_1(\epsilon > t) < \infty,$$

we need first to consider the homogeneous growth-fragmentation which corresponds to taking $\alpha = 0$. Specifically, we write $X^{(0)}(t) = \exp(\xi(t))$ for the self-similar Markov process with characteristics $(\Psi, 0)$, and then $\mathcal{X}^{(0)}$ and $\mathbf{X}^{(0)}$ for the corresponding cell-system and homogeneous growth-fragmentation. It has been shown in the proof of Corollary 3 in [5] that the growth-fragmentations \mathbf{X} and $\mathbf{X}^{(0)}$ can be constructed simultaneously, so that for every $q > 0$ with $\kappa(q) < 0$, there is the following upper bound for the extinction time of the growth-fragmentation \mathbf{X} :

$$\epsilon \leq c(q) S^{-\alpha/q},$$

where $c(q) > 0$ is some constant,

$$S := \sup \left\{ e^{-t\kappa(q)} (X_*^{(0)}(t))^q : t \geq 0 \right\},$$

and $X_*^{(0)}(t)$ denotes the size of the largest cell in the family $\mathbf{X}^{(0)}(t)$.

Recall from Proposition 3 of [5] that $\mathbf{X}^{(0)}(t) = (X_1^{(0)}(t), \dots)$ can be viewed as a compensated-fragmentation process (see Definition 3 in [4]), and then from Corollary 3 in [4] (it is assumed in [4] that $q \geq 2$, but actually only $\kappa(q) < \infty$ is needed) that the process

$$\exp(-t\kappa(q)) \sum_{i=1}^{\infty} (X_i^{(0)}(t))^q, \quad t \geq 0$$

is a martingale. We deduce from Doob's maximal inequality that $\mathbb{P}(S > x) = O(x^{-1})$, and thus for some constant $c < \infty$, we have $\mathbb{P}_1(\epsilon > t) \leq ct^{q/\alpha}$ for all $t \geq 0$, and then by self-similarity, that

$$\mathbb{P}_x(\epsilon > t) \leq cx^q t^{q/\alpha}.$$

To complete the proof, we simply observe from an application of the branching property of growth-fragmentation processes and the upper-bound above, that

$$\mathbb{P}_1(\epsilon > 2t) = \mathbb{E}_1(\epsilon > 2t \mid \mathcal{F}_t) \leq c\mathbb{E}_1 \left(\sum_{i=1}^{\infty} X_i^q(t) \right) \mathbb{P}_1(\epsilon > t) \leq cc(q)t^{(\omega_+ - q)/\alpha} t^{q/\alpha},$$

where the second inequality uses (16). □

Remark. We conjecture that, as a matter of facts,

$$\lim_{t \rightarrow \infty} t^{-\omega_+/\alpha} \mathbb{P}_1(\epsilon > t) = \mathbb{E}_0^+(1/M^+(1)),$$

but we do not have a rigorous proof of this. We note further that the arguments above show rather indirectly that $\mathbb{E}_0^+(1/M^+(t)) \in (0, \infty)$ for all $t > 0$, a fact which does not seem to follow straightforwardly from the construction of \mathcal{Q}_0^+ . More precisely, it is then easy to deduce that the process $(1/M^+(t) : t > 0)$ is a \mathbb{P}_0^+ supermartingale.

4.3 Tagging a cell randomly according to the intrinsic area

Throughout this section, we assume that Cramér's hypothesis (9) holds. Similarly to Sec. 4.1, but with simplifications due to the uniform integrability of \mathcal{M}^- (see Lemma 2.4), we introduce the probability measure $\widehat{\mathcal{P}}_x^-$ describing the joint law of a tagged leaf \mathcal{L} on $\partial\mathbb{U}$ and a cell system $\mathcal{X} = (\mathcal{X}_u : u \in \mathbb{U})$. The law of $(\mathcal{X}_u : u \in \mathbb{U})$ under $\widehat{\mathcal{P}}_x^-$ is absolutely continuous with respect to \mathcal{P}_x with density $x^{-\omega_-} \mathcal{M}^-(\infty)$, and conditionally on $(\mathcal{X}_u : u \in \mathbb{U})$, the random leaf \mathcal{L} has the law $\mathcal{A}(\cdot)/\mathcal{M}^-(\infty)$. Theorem 3.10 entails in particular that for $\alpha \geq 0$, the distribution \mathbb{P}_x^- of the growth-fragmentation process \mathbf{X} under $\widehat{\mathcal{P}}_x^-$ is absolutely continuous with respect to \mathbb{P}_x , with density $x^{-\omega_-} M^-(\infty)$. We use the same notation as that introduced in Sec. 4.1 and state without proof analogous results.

Proposition 4.6. *For every $t \geq 0$, every measurable function $f : [0, \infty) \rightarrow [0, \infty)$, and every $\bar{\mathcal{F}}_t$ -measurable random variable $B(t) \geq 0$, we have*

$$\mathcal{E}_x \left(\sum_{i=1}^{\infty} X_i^{\omega_-}(t) f(X_i(t)) B(t) \right) = x^{\omega_-} \widehat{\mathcal{E}}_x^-(f(\hat{\mathcal{X}}(t)) B(t)),$$

with the convention that $f(\partial) = 0$.

In turn, the spinal decomposition has now the following form. The function $\Phi^-(q) := \kappa(q + \omega_-)$ is the Laplace exponent of a Lévy process η^- , whose Lévy measure can be expressed in the form $e^{\omega_-x}(\Lambda(dx) + \tilde{\Lambda}(dx))$. This enables us to mark the negative jumps of η^- just as in Sec. 4.1, and the number $N^-(t)$ of marked jumps up to time t is a Poisson process with intensity $-\Psi(\omega_-)$. We then introduce the law \mathcal{Q}_x^- of the self-similar Markov process Y^- with characteristics (Φ^-, α) started from x . In the obvious notation (in particular $I^- := \int_0^\infty \exp(-\alpha\eta^-(s))ds$ and τ_t^- refers to the Lamperti's time substitution in this setting), we arrive at:

Theorem 4.7. *The distribution of $(\hat{\mathcal{X}}(t), n_t)_{0 \leq t < b_{\mathcal{L}}}$ under $\hat{\mathcal{P}}_1^-$ is the same as that of*

$$(Y^-(t), N^-(\tau_t^-))_{0 \leq t < I^-}.$$

Further, conditionally on $(\hat{\mathcal{X}}(t), n_t)_{0 \leq t < b_{\mathcal{L}}}$, the processes $\hat{\mathbf{X}}_{n,j}$ for $n \geq 0$ and $j \geq 1$ are independent, and more precisely, each $\hat{\mathbf{X}}_{n,j}$ has the (conditional) law \mathbb{P}_x , with x the absolute size of the negative jump of $\hat{\mathcal{X}}$ with label (n, j) .

An important difference compared to Sec. 4.1 is that now the Lévy process η^- drifts to $-\infty$, so Y^- is absorbed at ∂ after an a.s. finite time if $\alpha < 0$ (and more precisely, $Y^-(\zeta-) = 0$ \mathcal{Q}_x^- -a.s.), or has an infinite lifetime and converges to 0 at infinity if $\alpha \geq 0$. In both cases, we can rank the negative jumps of Y^- in the decreasing order, and introduce the law \mathcal{Q}_x^- of the cell system in which the Eve cell evolves according to Y^- and all the other cells according to X . The spinal decomposition then shows that the distribution of the growth-fragmentation \mathbf{X} under \mathcal{Q}_x^- is \mathbb{P}_x^- (recall that when $\alpha \geq 0$, the latter is absolutely continuous with respect to \mathbb{P}_x , with density $x^{-\omega_-} M^-(\infty)$).

5 A distinguished family of growth-fragmentations

The goal of this section is twofold. First, we give different ways to characterize the law of a self-similar growth-fragmentation. Recall that by [33], the law of a self-similar growth-fragmentation is characterized by a pair (κ, α) where κ is the cumulant function associated with a driving Lévy process and $\alpha \in \mathbb{R}$ is the self-similarity parameter. Theorem 5.1 shows that the cumulant function κ is characterized by one of the following quantities:

- any shift of the cumulant function κ , that is any function $q \mapsto \kappa(\omega + q)$ for fixed $\omega \geq 0$. As a consequence, the law of the process Y^+ (or of Y^-) introduced in Sec. 4.1 (resp. Sec. 4.3) characterizes the law of a self-similar growth-fragmentation;
- a particular cell process, which roughly speaking describes the evolution of the size of the locally largest fragment (that is the cell obtained by following the largest fragment at each splitting).

These are analytical results of independent interest.

Using this, we identify a remarkable one-parameter class of cumulant functions $\{\kappa_\theta : \theta \in (1/2, 3/2]\}$ which are closely related to stable Lévy processes as well as to random maps (see Sec. 6 for the latter connection). More precisely, for every $\theta \in (\frac{1}{2}, \frac{3}{2}]$, we show the existence of a driving Lévy process (see (28) for the expression of its Laplace exponent) such that the associated cumulant function κ_θ is

$$\kappa_\theta(q) = \frac{\cos(\pi(q - \theta))}{\sin(\pi(q - 2\theta))} \cdot \frac{\Gamma(q - \theta)}{\Gamma(q - 2\theta)}, \quad \theta < q < 2\theta + 1. \quad (19)$$

In the notation of Sec. 3, κ_θ satisfies Cramér's condition with

$$\omega_- = \theta + 1/2, \quad \omega_+ = \theta + 3/2 \quad (20)$$

and we will see below that $\Phi^-(q) = \kappa(\omega_- + q)$ and $\Phi^+(q) = \kappa(\omega_+ + q)$ are, up to scaling constants, the Laplace exponents of the Lévy process appearing in the Lamperti representation of a strictly θ -stable Lévy process with positivity parameter ρ conditioned to die continuously at 0, resp. to stay positive, and such that

$$\theta(1 - \rho) = 1/2. \quad (21)$$

We refer to [18, 17] for a definition of these processes. In other words, the process Y^+ and Y^- defined in Sec. 4.1, 4.3 and associated with the self-similar growth-fragmentation characterized by the pair $(\kappa_\theta, -\theta)$ are distributed as the θ -stable Lévy processes with positivity parameter ρ conditioned to stay positive, resp. conditioned to die continuously at 0.

In the particular case $\theta = 3/2$, note that there are no positive jumps and $\kappa_{3/2}(q) = \frac{\Gamma(q-3/2)}{\Gamma(q-3)}$ so that, up to a scaling constant, the growth-fragmentation characterized by the pair $(\kappa_{3/2}, -1/2)$ is the one that has been considered in [6] (see in particular Eq. (32) in [6]). Using the connection with random maps established in Sec. 6 we will be able to identify the law of the intrinsic area of these growth-fragmentations with cumulant function κ_θ , which turn out to be size-biased stable distributions (Corollary 6.9).

5.1 Characterizing the cumulant function of a growth-fragmentation

If κ is a cumulant function of a growth-fragmentation and satisfies (7), we have seen in Lemma 3.4 that $q \mapsto \kappa(w + q)$ is the Laplace exponent of a (possibly killed) Lévy process provided that $\kappa(w) \leq 0$ (this lemma is actually stated for $w = \omega_+$, but the argument is the same). Conversely, if Φ is the Laplace exponent of a (possibly killed) Lévy process, it is natural to ask if there exists a growth-fragmentation with cumulant function κ and a value w such that $\Phi(\cdot) = \kappa(w + \cdot)$, and in this case if w is uniquely determined. It turns out that such a growth-fragmentation does not always exist, and we now provide a necessary and sufficient condition for this to hold.

Theorem 5.1. *Let Φ be a Laplace exponent of a (possibly killed) Lévy process written in the form*

$$\Phi(q) = \Phi(0) + \frac{1}{2}s^2q^2 + b'q + \int_{\mathbb{R}} (e^{qy} - 1 + q(1 - e^y)) \Lambda(dy), \quad q \geq 0, \quad (22)$$

where $\sigma^2 \geq 0$, $b \in \mathbb{R}$, $\Phi(0) \leq 0$ and Λ is a measure on \mathbb{R} such that $\int (1 \wedge y^2) \Lambda(dy) < \infty$ and² $\int_{y>1} e^{qy} \Lambda(dy) < \infty$. The following two assertions are equivalent:

- (i) *There exists a unique pentuple $(\sigma^2, b, \mathbf{k}, w, \Lambda_L)$ with $\sigma^2 \geq 0$, $b \in \mathbb{R}$, $\mathbf{k} \geq 0$, $w \geq 0$ and Λ_L a measure on $(-\ln(2), \infty)$ with $\int_{(-\ln(2), \infty)} (1 \wedge y^2) \Lambda_L(dy) < \infty$ such that we have*

$$\kappa(w + q) = \Phi(q), \quad \forall q \geq 0,$$

where we have put

$$\kappa(q) = -\mathbf{k} + \frac{1}{2}\sigma^2q^2 + bq + \int_{(-\ln(2), \infty)} (e^{qy} - 1 + (1 - e^y)^q \mathbb{1}_{y<0} + q(1 - e^y)) \Lambda_L(dy). \quad (23)$$

- (ii) *Let ν be the image of the measure Λ by the map $x \mapsto e^x$. There exists a unique $w \geq 0$ such that the measure $x^{-w}\nu(dx)$, restricted to $(0, 1)$, is symmetric with respect to $1/2$. Set*

$$f(q) = \Phi(q - w) - \int_{(0, \infty)} (e^{qy} - 1 + q(1 - e^y)) e^{-wy} \Lambda(dy). \quad (24)$$

²The condition $\int_{y>1} e^{qy} \Lambda(dy) < \infty$ may be replaced by the weaker condition that there exists $q > 0$ such that $\int_{y>1} e^{qy} \Lambda(dy) < \infty$, and by considering an additional cutoff in (22) but we shall not enter such considerations.

Then $f(q)$ is well defined and finite for every $q \in [1, w + 1]$, the function f admits an analytic continuation to $[1, +\infty)$, and we have

$$15f(4) \geq 6f(5) + 10f(3). \quad (25)$$

In addition, when these assertions are satisfied, $s^2 = \sigma^2$, $\Lambda_L(dy) = e^{-wy}\Lambda(dy)$ on $(-\ln(2), \infty)$ and

$$\mathbf{k} = 15f(4) - 6f(5) - 10f(3).$$

In particular, $\mathbf{k} = 0$ if and only if the inequality in (25) is an equality.

Given a growth-fragmentation with cumulant function κ , Theorem 5.1 (and more precisely (23)) shows that the self-similar Markov process associated with the (possibly killed) Lévy process with Laplace exponent

$$q \longmapsto -\mathbf{k} + \frac{1}{2}\sigma^2 q^2 + bq + \int_{(-\ln(2), \infty)} (e^{qy} - 1 + q(1 - e^y)) \Lambda_L(dy) \quad (26)$$

may be used as the cell process to construct the growth-fragmentation. Roughly speaking, this self-similar Markov process describes the evolution of the size of the locally largest fragment (that is the cell obtained by following the largest fragment at each splitting, indeed $X(t) \geq X(t-)/2$ for every $t > 0$ for this self-similar Markov process X) in the growth-fragmentation. In particular, Theorem 5.1 gives a means to analytically identify this process.

Theorem 5.1 also implies that the laws of the processes Y^+ and Y^- introduced in Sec. 4.1 and 4.3 characterize the law of a self-similar growth-fragmentation.

Proof. We first show that (i) implies (ii). Let $(\sigma^2, b, \mathbf{k}, w, \Lambda_L)$ be a pentuple such that (i) holds, and let ν_L be the image of the measure Λ_L by the map $x \mapsto e^x$. Introduce the symmetrized measure $\bar{\nu}_L$ of ν_L on $(0, 1)$ defined by

$$\int_{(0,1)} g(x) \bar{\nu}_L(dx) = \int_{(1/2,1)} (g(x) + g(1-x)) \nu_L(dx)$$

for every nonnegative measurable function g . By using the definition of κ , a straightforward computation yields the equality

$$\kappa(w+q) = \kappa(w) + \frac{1}{2}\sigma^2 q^2 + b_1 q + \int_{(0,1)} (x^q - 1 + q(1-x)) x^\omega \bar{\nu}_L(dx) + \int_{(1,\infty)} (x^q - 1 + q(1-x)) x^\omega \nu_L(dx)$$

for every $q \geq 0$, with a certain $b_1 \in \mathbb{R}$. Since by hypothesis $\kappa(w+q) = \Phi(q)$ for every $q \geq 0$, the Lévy-Khintchin formula implies that $\kappa(w) = \Phi(0)$, $x^\omega \bar{\nu}_L(dx) = \nu(dx)$ on $(0, 1)$ and that $x^\omega \nu_L(dx) = \nu(dx)$ on $(1, \infty)$. In particular, on $(0, 1)$, $x^{-\omega} \nu(dx)$ is symmetric with respect to $1/2$.

We now check that

$$15f(4) - 6f(5) - 10f(3) = \mathbf{k}$$

and (25) will follow. To this end, first notice $f(q) < \infty$ for every $q \in [1, w + 1]$ since $\int_{y>1} e^y \Lambda(dy) < \infty$. In addition, we have

$$\begin{aligned} f(q) &= \kappa(q) - \int_{(0,\infty)} (e^{qy} - 1 + q(1 - e^y)) \Lambda_L(dy) \\ &= -\mathbf{k} + \frac{1}{2}\sigma^2 q^2 + bq + \int_{\mathbb{R}_-} (e^{qy} + (1 - e^y)^q - 1 + q(1 - e^y)) \Lambda_L(dy). \end{aligned}$$

so that f admits indeed an analytic continuation on $[1, \infty)$. Set $P_q(x) = x^q + (1-x)^q - 1 + q(1-x)$ for every $q, x \in \mathbb{R}$. Since for every $x \in \mathbb{R}$ we have $2P_3(x) - 3P_2(x) = 0$ and $5P_4(x) - 2P_5(x) - 5P_2(x) = 0$, it follows that

$$2f(3) - 3f(2) = \mathbf{k} + 3\sigma^2 \quad \text{and} \quad 5f(4) - 2f(5) - 5f(2) = 2\mathbf{k} + 5\sigma^2.$$

As a consequence, $15f(4) - 6f(5) - 10f(3) = \mathbf{k}$, and the proof of the first implication is complete.

Finally, if (ii) is satisfied, the previous calculations show that $\kappa(q) := \Phi(q-w)$ can be written in the form (23) with $\sigma^2 = s^2$, $\Lambda_L(dy) = e^{-wy}\Lambda(dy)$ on $(-\ln(2), \infty)$, $\mathbf{k} = 15f(4) - 6f(5) - 10f(3)$ and a certain value of $b \in \mathbb{R}$. \square

5.2 A one-parameter family of cumulant functions

Recall that if κ is defined by (5) and satisfies (7), we have seen in Lemma 3.4 that Φ^+ is the Laplace exponent of a Lévy process drifting to $+\infty$. As an application of Theorem 5.1, we exhibit a distinguished family of cumulant functions for which the Laplace exponent Φ^+ (and also Φ^-) takes a particularly simple form. This sheds some new light on a calculation performed by Miller & Sheffield [31, Sec. 4] (see the Remark 5.3 at the end of this section). Let us mention that the proof of Proposition 5.2 is technical since it uses hypergeometric Lévy processes [26] (and hence hypergeometric functions) and may be skipped in first reading.

Proposition 5.2. *Assume that Φ^+ is the Laplace exponent of a hypergeometric Lévy process without killing and drifting to $+\infty$, that is*

$$\Phi^+(z) = -\frac{\Gamma(\gamma - z)}{\Gamma(-z)} \cdot \frac{\Gamma(\widehat{\beta} + \widehat{\gamma} + z)}{\Gamma(\widehat{\beta} + z)}. \quad (27)$$

with $\widehat{\beta} > 0$, $\gamma, \widehat{\gamma} \in (0, 1)$. Then the assertions of Theorem 5.1 are satisfied if and only if $\widehat{\beta} = 1$ and $\widehat{\gamma} \in (0, 1/2]$. Then Φ^+ is the Laplace exponent of the Lévy process appearing in the Lamperti representation of a strictly θ -stable Lévy process with positivity parameter ρ conditioned to stay positive with $\theta = \gamma + \widehat{\gamma} \in (0, 3/2]$ and $\rho = \gamma/\theta$.

In addition, the associated growth-fragmentation has no killing (that is $\mathbf{k} = 0$ in (23)) if and only if $\theta(1 - \rho) = 1/2$ (in particular, $1/2 < \theta \leq 3/2$), and then its cumulant function is

$$\kappa_\theta(q) = \frac{\cos(\pi(q - \theta))}{\sin(\pi(q - 2\theta))} \cdot \frac{\Gamma(q - \theta)}{\Gamma(q - 2\theta)}, \quad \theta < q < 2\theta + 1.$$

Proof. Let ν be the image by the map $x \mapsto e^x$ of the Lévy measure of the Lévy process with Laplace exponent given by (27). By [26, Proposition 1], setting $\eta = \widehat{\beta} + \gamma + \widehat{\gamma}$, the density of ν on $(0, 1)$ is

$$\nu(z) = -\frac{\Gamma(\eta)}{\Gamma(\eta - \gamma)\Gamma(-\widehat{\gamma})} z^{\widehat{\beta} + \widehat{\gamma} - 1} {}_2F_1(1 + \widehat{\gamma}, \eta; \eta - \gamma; z), \quad \text{with} \quad {}_2F_1(a, b; c; z) = \sum_{n \geq 0} \frac{\Gamma(a + n)\Gamma(b + n)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(n + c)} z^n.$$

Let $w \geq 0$ be such that $z^{-w}\nu(z)$ is symmetric with respect to $1/2$ on $(0, 1)$. Then

$$\frac{{}_2F_1(1 + \widehat{\gamma}, \eta; \eta - \gamma; z)}{{}_2F_1(1 + \widehat{\gamma}, \eta; \eta - \gamma; 1 - z)} = \left(\frac{z}{1 - z}\right)^{w - \widehat{\beta} - \widehat{\gamma} + 1}.$$

Using the formula (see e.g. [37, p. 291])

$$\begin{aligned} {}_2F_1(a, b; c; z) &= \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} {}_2F_1(a, b; a + b + 1 - c; 1 - z) \\ &\quad + \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)} (1 - z)^{c - a - b} {}_2F_1(c - a, c - b; 1 + c - a - b; 1 - z) \end{aligned}$$

and taking $z \rightarrow 0$, it is a simple matter to check that this implies that $1 + \hat{\gamma} = \eta - \gamma$, so that $\hat{\beta} = 1$. Setting $\theta = \gamma + \hat{\gamma}$, this also yields $w = \theta + \hat{\gamma} + 1$ and ${}_2F_1(1 + \hat{\gamma}, \eta; \eta - \gamma; z) = (1 - z)^{-\theta-1}$. As a consequence, by [26, Proposition 1], the density of ν is

$$\nu(z) = \frac{\Gamma(\theta + 1) \sin(\pi\hat{\gamma})}{\pi} \frac{z^{\hat{\gamma}}}{(1 - z)^{\theta+1}} \mathbb{1}_{0 < z < 1} + \frac{\Gamma(\theta + 1) \sin(\pi\gamma)}{\pi} \frac{z^{\theta-\gamma}}{(z - 1)^{\theta+1}} \mathbb{1}_{z > 1},$$

so that by [26, Theorem 1], Φ^+ is indeed the Laplace exponent of the Lévy process appearing in the Lamperti representation of a strictly θ -stable Lévy process with positivity parameter $\rho = \gamma/\theta$ conditioned to stay positive.

It remains to see if (25) holds, and to this end we calculate $\int_{(1, \infty)} (x^q - 1 + q(1 - x)) x^{-w} \nu(dx)$ up to a linear term in q . On $(1, \infty)$, we have

$$z^{-w} \nu(dz) = \frac{\Gamma(\theta + 1) \sin(\pi\gamma)}{\pi} \frac{1}{(z(z - 1))^{\theta+1}} = \frac{\Gamma(\theta + 1) \sin(\pi\gamma)}{\pi} \frac{z^{-2\theta-2}}{(1 - z^{-1})^{\theta+1}}.$$

Since a Lévy process with Lévy measure $z^{-w} \nu(dz)$ on $(1, \infty)$ belongs to the so-called class of β -family of Lévy processes introduced by Kuznetsov [25], by [25, Proposition 9] (with, in the notation of the latter reference, $\beta_1 = 1$, $\lambda_1 = \theta + 1$, $\alpha_1 = 2\theta + 1$) we get the existence of $b \in \mathbb{R}$ such that

$$\int_{(1, \infty)} (x^q - 1 + q(1 - x)) x^{-w} \nu(dx) = -\frac{\sin(\gamma\pi)}{\sin(\theta\pi)} \left(\frac{\Gamma(2\theta + 1 - q)}{\Gamma(\theta + 1 - q)} - \frac{\Gamma(2\theta + 1)}{\theta + 1} \right) + bq, \quad 0 \leq q < 2\theta + 1.$$

As a consequence,

$$f(q) = \frac{\sin(\pi(\theta + \hat{\gamma} - q))}{\pi} \Gamma(2\theta + 1 - q) \Gamma(q - \theta) + \frac{\sin(\gamma\pi)}{\sin(\theta\pi)} \left(\frac{\Gamma(2\theta + 1 - q)}{\Gamma(\theta + 1 - q)} - \frac{\Gamma(2\theta + 1)}{\theta + 1} \right) - bq, \quad 0 \leq q < 2\theta + 1.$$

Since $\sigma^2 = 0$, the proof of Theorem 5.1 shows that $\mathbf{k} = 2f(3) - 3f(2)$. A straightforward computation then gives

$$\mathbf{k} = \cos(\hat{\gamma}\pi) \cdot \frac{2\Gamma(2\theta)}{\Gamma(\theta)}.$$

Therefore $\mathbf{k} \geq 0$ if and only if $\hat{\gamma} \in (0, 1/2]$, which implies that $\theta \in (0, 3/2]$. In addition, $\mathbf{k} = 0$ if and only if $\theta(1 - \rho) = \hat{\gamma} = 1/2$. \square

In the particular case of κ_θ (with $\theta \in (1/2, 3/2]$), the proof of Proposition 5.2 actually gives the explicit Laplace exponent (26) of the Lévy process involved in the self-similar Markov process describing the evolution of the locally largest fragment in the associated growth-fragmentation. Let ν_θ be the measure on $(1/2, \infty)$ defined by

$$\nu_\theta(dx) = \frac{\Gamma(\theta + 1)}{\pi} \left(\frac{1}{(x(1 - x))^{\theta+1}} \mathbb{1}_{1/2 < x < 1} + \sin(\pi(\theta - 1/2)) \cdot \frac{1}{(x(x - 1))^{\theta+1}} \mathbb{1}_{x > 1} \right) dx$$

and let Λ_θ be the image of ν_θ by the map $x \mapsto \ln x$. Set

$$\Psi_\theta(q) = \left(\frac{\Gamma(2 - \theta)}{2\Gamma(2 - 2\theta) \sin(\pi\theta)} + \frac{\Gamma(\theta + 1) B_{\frac{1}{2}}(-\theta, 2 - \theta)}{\pi} \right) q + \int_{\mathbb{R}} (e^{qy} - 1 + q(1 - e^y)) \Lambda_\theta(dy) \quad (28)$$

where $B_{1/2}(a, b) = \int_0^{1/2} t^{a-1} (1 - t)^{b-1} dt$ is the incomplete Beta function. Then

$$\kappa_\theta(q) = \Psi_\theta(q) + \int_{(-\infty, 0)} (1 - e^y)^q \Lambda_\theta(dy), \quad \theta < q < 2\theta + 1.$$

Indeed, the proof of Proposition 5.2 shows the existence of $b \in \mathbb{R}$ such that

$$\frac{\cos(\pi(q-\theta))}{\sin(\pi(q-2\theta))} \cdot \frac{\Gamma(q-\theta)}{\Gamma(q-2\theta)} = bq + \int_{1/2}^{\infty} (x^q - 1 + (1-x)^q \mathbb{1}_{x < 1} + q(1-x)) \nu_{\theta}(dx), \quad \theta < q < 2\theta + 1.$$

To find the value of b , we simply specify the latter equality for $q = 2$:

$$-\frac{\Gamma(2-\theta)}{2\Gamma(2-2\theta)\sin(\pi\theta)} = 2b + 2 \int_{1/2}^1 (1-x)^2 \nu_{\theta}(dx) + \int_1^{\infty} (1-x)^2 \nu_{\theta}(dx).$$

Since

$$\int_{1/2}^1 \frac{1}{x^{\theta+1}(1-x)^{\theta-1}} dx = -\frac{\Gamma(1-\theta)\Gamma(2-\theta)}{\theta\Gamma(2-2\theta)} - B_{\frac{1}{2}}(-\theta, 2-\theta)$$

and

$$\int_1^{\infty} \frac{1}{x^{\theta+1}(x-1)^{\theta-1}} dx = \frac{\Gamma(2-\theta)\Gamma(2\theta-1)}{\Gamma(\theta+1)},$$

we obtain the announced value of b .

In particular, by taking $\theta = 3/2$, since $B_{\frac{1}{2}}(-3/2, 1/2) = -8/3$, we get that

$$\frac{\Gamma(q-3/2)}{\Gamma(q-3)} = -\frac{2}{\sqrt{\pi}}q + \frac{3}{4\sqrt{\pi}} \int_{1/2}^1 (x^q - 1 + q(1-x) + (1-x)^q) \cdot \frac{1}{(x(1-x))^{5/2}} dx \quad (q > 3/2),$$

which gives a proof of the identity [6, Eq. (32)].

Remark 5.3 (Interpretation of a calculation by Miller & Sheffield). Let us draw a connection between this section and a calculation performed by Miller & Sheffield [31, Sec. 4]. We first translate (non-rigorously) their setup in our framework of growth-fragmentations³: Consider a growth-fragmentation with pair (κ, α) such that Cramér's hypothesis holds. Assume that the process Y^- (in the notation of Sec. 4.3) is the time-reversal of a θ -stable branching process. We claim that necessarily we have $\theta = 3/2$. Indeed, if Y^- is a time-reversal of the θ -stable branching process, by standard reversal arguments Y^- is a Lamperti time-change of the θ -stable process with no positive jumps conditioned to die continuously at 0. Therefore by Theorem 5.1 we must have $\kappa = \kappa_{\theta}$ and the only θ for which the process has no positive jumps is $\theta = 3/2$.

Remark 5.4. The boundary case $\theta = 1/2$ and $\rho = 0$ corresponds to $\kappa_{1/2}(q) = -\frac{\Gamma(q-1/2)}{\Gamma(q-1)}$, which is (up to a constant factor) the cumulant function of the self-similar pure fragmentation that occurs when splitting the Brownian Continuum Random Tree at heights; see [3] and [36]. Roughly speaking, in this situation, we have $\omega_- = 1$ whereas ω_+ does not exist, as $\kappa_{1/2}$ is a non-increasing function on $[1, \infty)$. Further, the function $\Phi_{1/2}^-(q) = \kappa_{1/2}(q+1)$ can be identified as the Laplace exponent of the Lévy process appearing in the Lamperti representation of the negative of a $1/2$ -stable subordinator killed when it becomes negative, and conditioned to hit 0; see [36] for details. Note further that in this case, it would make of course no sense to condition this process to stay positive.

³More precisely, keeping the notation of [31, Sec. 4], Theorem 4.6 in [31] indicates that under $\tilde{\mu}_{\text{DISK}}^{1,L}$, the process describing the boundary lengths of increasing balls from the root is a self-similar growth-fragmentation under the tilted probability measure \mathbb{P}_L^- . The process (L_r) is Y^- (the size of the tagged fragment under $\widehat{\mathcal{P}}_L^-$), the process (M_r^1) is the size of the Eve cell when one uses the locally largest cell process for the Eve cell under the tilted probability measure $\widehat{\mathcal{P}}_L^-$ and the process (M_r) is the size of the Eve cell when one uses the locally largest cell process for the Eve cell under the non-tilted probability measure $\widehat{\mathcal{P}}_L$. Theorem 4.6 in [31] indicates that the process (L_r) evolves as a time-reversed θ -stable continuous state branching process.

6 Applications to large random planar maps

In this section, we show that growth-fragmentations with cumulant function (19) appear in the Markovian explorations of particular models of random planar maps which are, roughly speaking, the dual maps of the so-called stable maps of Le Gall & Miermont [28]. In particular, for $\theta \in (1, 3/2]$, we shall show that when taken with the proper self-similarity index, they describe the scaling limit of the perimeters of cycles obtained by slicing these random maps at all heights. For $\theta = 3/2$, this was observed in [6] for random Boltzmann triangulations. In another direction, this link will allow us to identify the law of the intrinsic area of these growth-fragmentations (as defined in Sec. 4.3).

6.1 Critical non-generic Boltzmann planar maps

We first present the model of random planar maps we are dealing with. As usual, all planar maps in this work are rooted, i.e. come with a distinguished oriented edge; for technical simplicity we will only consider *bipartite* planar maps, that is all faces have even degree. If \mathbf{m} is a (rooted bipartite) planar map we denote by $\text{Faces}(\mathbf{m})$ the set of its faces, and by $f_r \in \text{Faces}(\mathbf{m})$ the face adjacent to the right of the root edge. This face is called the root face of the map, and the origin vertex of the root edge is called the origin of the map. The integer $\deg(f_r)$ is the perimeter of \mathbf{m} ; note that the perimeter of a bipartite map must be even due to the parity constraint. We write $|\mathbf{m}|$ for the total number of vertices of \mathbf{m} . For $\ell \geq 0$ and $n \geq 0$, we denote by $\text{Map}_n^{(\ell)}$ the set of all (rooted bipartite) planar maps of perimeter 2ℓ with n vertices. By convention, $\text{Map}_1^{(0)}$ contains a single “vertex map”. We finally set $\text{Map}^{(\ell)} = \bigcup_{n \geq 0} \text{Map}_n^{(\ell)}$. Any planar map with at least one edge can be seen as a planar map with perimeter 2 by simply splitting the root edge into a root face of degree 2. We shall implicitly make this identification many times in this section.

Given a non-zero sequence $\mathbf{q} = (q_k)_{k \geq 1}$ of non-negative real numbers, we define a measure \mathbf{w} on the set of all (finite) bipartite planar maps by the formula

$$\mathbf{w}(\mathbf{m}) := \prod_{f \in \text{Faces}(\mathbf{m}) \setminus \{f_r\}} q_{\deg(f)/2}, \quad \mathbf{m} \in \bigcup_{\ell \geq 0} \text{Map}^{(\ell)}.$$

We then set

$$W_n^{(\ell)} = \mathbf{w}(\text{Map}_n^{(\ell)}), \quad W^{(\ell)} = \sum_{n \geq 0} W_n^{(\ell)} \quad \text{and} \quad W_{\bullet}^{(\ell)} = \sum_{n \geq 0} n W_n^{(\ell)}, \quad (29)$$

where the dependence in \mathbf{q} is implicit. We assume that \mathbf{q} is admissible, meaning that $W_{\bullet}^{(\ell)} < \infty$ for one value of $\ell \geq 1$ (or, equivalently, that $W_{\bullet}^{(\ell)} < \infty$ for every $\ell \geq 1$, see e.g. [15]). As in [28, Sec. 2.2] and in [14], we henceforth focus on the case where the admissible weight sequence \mathbf{q} is critical and non-generic, which in particular implies that

$$q_k \underset{k \rightarrow \infty}{\sim} c \gamma^{k-1} k^{-\theta-1}, \quad \text{for a certain } \theta \in \left(\frac{1}{2}, \frac{3}{2}\right) \text{ and } c, \gamma > 0. \quad (30)$$

The reader should keep in mind that the above weight sequence \mathbf{q} must be very fine-tuned to achieve non-generic criticality, see [28, 15, 14]. To avoid these complications, one may decide to work with the following concrete admissible, critical and non-generic weight sequence for fixed $\theta \in (\frac{1}{2}, \frac{3}{2})$, see [16, Sec. 5]:

$$q_k = c \gamma^{k-1} \frac{\Gamma(k - \theta - \frac{1}{2})}{\Gamma(\frac{1}{2} + k)} \mathbb{1}_{k \geq 2}, \quad \gamma = \frac{1}{4\theta + 2}, \quad c = \frac{-\sqrt{\pi}}{2\Gamma(\frac{1}{2} - \theta)}. \quad (31)$$

Note that the values a and κ in [15, 16] are denoted here by respectively $\theta + 1$ and γ .

We denote by $\mathbb{P}^{(\ell)}$ the probability measure $\mathbf{w}(\cdot \mid \cdot \in \text{Map}^{(\ell)})$ and we say that a random map with distribution $\mathbb{P}^{(\ell)}$ is a \mathbf{q} -Boltzmann planar map with perimeter 2ℓ . Under our assumptions on \mathbf{q} , the scaling limit of these maps under $\mathbb{P}^{(1)}$ conditioned on having a fixed large number of vertices, is given (at least along subsequences) by the so-called stable maps of Le Gall & Miermont [28] (which are random compact metric spaces that look like randomized versions of the Sierpinski carpet or gasket).

We shall also consider pointed \mathbf{q} -Boltzmann planar maps. By definition, a pointed map is a pair (\mathbf{m}, v) where \mathbf{m} is a planar map and $v \in \mathbf{m}$ is a vertex (which is called the distinguished vertex). We denote by $\mathbb{P}_{\bullet}^{(\ell)}$ the probability distribution of the set of all pointed planar maps given by

$$\mathbb{P}_{\bullet}^{(\ell)}((\mathbf{m}, v)) = \frac{\mathbf{w}(\mathbf{m})}{W_{\bullet}^{(\ell)}}, \quad \mathbf{m} \in \text{Map}^{(\ell)}, v \in \mathbf{m}. \quad (32)$$

We say that a random variable with distribution $\mathbb{P}_{\bullet}^{(\ell)}$ is a pointed \mathbf{q} -Boltzmann planar map with perimeter 2ℓ . If $(B_{\bullet}^{(\ell)}, v_{\bullet})$ is such a random variable, note that for every $\mathbf{m} \in \text{Map}^{(\ell)}$, we have $\mathbb{P}(B_{\bullet}^{(\ell)} = \mathbf{m}) = |\mathbf{m}| \mathbf{w}(\mathbf{m}) / W_{\bullet}^{(\ell)}$, so a pointed \mathbf{q} -Boltzmann planar map is a size-biased \mathbf{q} -Boltzmann planar map with perimeter 2ℓ . Under our assumptions on \mathbf{q} , the degree of a typical face of a pointed \mathbf{q} -Boltzmann planar map is in the domain of attraction of a stable law of index $\theta + 1/2$ (see [28, Sec. 3.2])

It is further possible to define an infinite version of a \mathbf{q} -Boltzmann planar map with perimeter 2ℓ as the local limit of \mathbf{q} -Boltzmann planar maps with perimeter 2ℓ conditioned to have size tending to ∞ , see [35, Theorem 6.1]. The law of this infinite version is denoted by $\mathbb{P}_{\infty}^{(\ell)}$, which is a probability measure on the set of all infinite (bipartite) planar maps with perimeter 2ℓ .

Enumeration. We now recall some important enumeration results (we refer to [15] and [14] for proofs, see also [16] where they have been gathered under the present form). First, recalling that \mathbf{q} is an admissible, critical and non-generic weight sequence such that (30) holds, we read from [14, Eq. 3.15, Eq. 3.16] that

$$W^{(\ell)} \underset{\ell \rightarrow \infty}{\sim} \frac{c}{2 \cos((\theta + 1)\pi)} \gamma^{-\ell-1} \ell^{-\theta-1}. \quad (33)$$

The asymptotic behavior of $W_{\bullet}^{(\ell)}$ can be deduced from the following surprisingly universal identity (see [15, Eq. (17)])

$$\gamma^{\ell} W_{\bullet}^{(\ell)} = h^{\triangleright}(\ell) := 2^{-2\ell} \binom{2\ell}{\ell}, \quad \ell \geq 1. \quad (34)$$

We shall also need to estimate the asymptotic behavior of the partition functions of size-constrained Boltzmann planar maps. The following result appears in [14, Sec. 3.3] for $\ell = 1$, but we give a proof in the general case for completeness.

Proposition 6.1. *For every fixed $\ell \geq 1$,*

$$W_n^{(\ell)} \underset{n \rightarrow \infty}{\sim} c_0 \cdot h^{\uparrow}(\ell) \cdot \gamma^{-\ell} \cdot n^{-\frac{4(\theta+1)}{2\theta-1}}, \quad \text{with } c_0 = \frac{1}{2|\Gamma(-\frac{1}{\theta+1/2})|} \cdot \left(\frac{2\gamma \cos(\pi(\theta+1))\Gamma(\theta+3/2)}{c\sqrt{\pi}} \right)^{\frac{1}{\theta+1/2}}.$$

Proof. Note that, for every $g \in [0, 1]$,

$$\sum_{n \geq 0} n W_n^{(\ell)} g^n = \sum_{\mathbf{m} \in \text{Map}^{(\ell)}, v \in \mathbf{m}} \mathbf{w}(\mathbf{m}) g^{|\mathbf{m}|},$$

and that this quantity is equal to $W_{\bullet}^{(\ell)}$ for $g = 1$. The discussions just before and after Eq. (24) in [16] show that

$$\sum_{n \geq 0} n W_n^{(\ell)} g^n = g \cdot x(g)^\ell \cdot W_{\bullet}^{(\ell)}, \quad (35)$$

where $x(g) \in (0, 1]$ is the unique solution of

$$g = F(x(g)), \quad \text{where} \quad F(x) = \frac{x}{4\gamma} \left(1 - \sum_{k=1}^{\infty} \left(\frac{x}{4\gamma} \right)^{k-1} \binom{2k-1}{k} q_k \right).$$

Equivalently, we have

$$x(g) = g \cdot \Phi(x(g)), \quad \text{with} \quad \Phi(z) = \frac{z}{F(z)}.$$

By (30), using [22, VI.19 p407] we have that $F(z) = 1 - \frac{c\Gamma(-1/2-\theta)}{2\gamma\sqrt{\pi}}(1-z)^{\theta+1/2} + o((1-z)^{\theta+1/2})$ as $z \rightarrow 1, |z| < 1$, within a cone, so that

$$\Phi(z) = 1 + \frac{c\Gamma(-1/2-\theta)}{2\gamma\sqrt{\pi}}(1-z)^{\theta+1/2} + o((1-z)^{\theta+1/2}) \text{ as } z \rightarrow 1, |z| < 1, \text{ within a cone.}$$

By [22, VI.18 p407], it follows that

$$[g^n]x(g) \underset{n \rightarrow \infty}{\sim} \frac{1}{|\Gamma(-\frac{1}{\theta+1/2})|} \cdot \left(\frac{2\gamma\sqrt{\pi}}{c\Gamma(-1/2-\theta)} \right)^{\frac{1}{\theta+1/2}} \frac{1}{n^{\frac{1}{\theta+1/2}+1}}.$$

The result readily follows from (35). \square

Proposition 6.1 implies that for every $v \in \mathbb{Z}, \ell \geq 1$

$$\frac{\gamma^\ell W_{n-v}^{(\ell)}}{\gamma W_n^{(1)}} \underset{n \rightarrow \infty}{\longrightarrow} h^\uparrow(\ell) := 2\ell \cdot 2^{-2\ell} \binom{2\ell}{\ell}. \quad (36)$$

This convergence (36) has been previously established (in the more general setting where also faces of odd degree are allowed) in [15, Corollary 2] in a rather indirect way.

Harmonic functions. The functions h^\succ and h^\uparrow , which, remarkably, do not depend on the weight sequence \mathbf{q} will play an important role below, in particular through their relation with a random walk whose step distribution we now define. Let ν be the probability measure on \mathbb{Z} defined by

$$\nu(k) = \begin{cases} q_{k+1}\gamma^{-k} & \text{for } k \geq 0 \\ 2W^{(-1-k)}\gamma^{-k} & \text{for } k \leq -1. \end{cases} \quad (37)$$

Under our assumptions, ν is indeed a probability distribution which is centered and in the domain of attraction of the θ -stable law with positivity parameter ρ , and which further satisfies (21). Let $(S_n)_{n \geq 0}$ be the random walk on \mathbb{Z} with independent increments distributed according to ν . It has been remarked in [15] that h^\uparrow , with the convention $h^\uparrow(\ell) = 0$ when $\ell \leq 0$, is up to a multiplicative constant the unique harmonic function on $\{1, 2, 3, \dots\}$ for this random walk (we say that h^\uparrow is ν -harmonic at these points) that vanishes on $\{\dots, -2, -1, 0\}$. This fact has been used in [15] to give an alternative definition of critical weight sequences (see Corollary 1 in [15]). We also note that h^\succ is the discrete derivative of the function h^\uparrow , namely

$$h^\succ(\ell) = h^\uparrow(\ell+1) - h^\uparrow(\ell), \quad \ell \geq 0.$$

It readily follows that h^\succ is harmonic on $\{1, 2, 3, \dots\}$ and vanishes on $\{\dots, -2, -1\}$. By classical results [7] we deduce that the h -transform of the walk (S_n) with the harmonic functions h^\uparrow (resp. h^\succ) can be interpreted as the walk (S_n) conditioned to stay positive (resp. to be absorbed at 0 without touching \mathbb{Z}^-). This fact should be reminiscent of the various transformations performed in Sec. 4.

6.2 Edge-peeling explorations

As we said above, the growth-fragmentation processes of Sec. 5.2 with cumulant function κ_θ will appear as a scaling limit of the perimeters of the holes encountered in the Markovian explorations of \mathbf{q} -Boltzmann random planar maps. By exploration we mean the so-called “lazy peeling process” introduced in [15]. We first present the branching peeling exploration (using the presentation of [16]) in a deterministic setting and move to random maps in the next section.

6.2.1 Submaps in the primal and dual maps

Let \mathbf{m} be a (bipartite rooted) planar map. We denote by \mathbf{m}^\dagger the rooted (not necessarily bipartite) map which is the dual map of \mathbf{m} , whose vertices are the faces of \mathbf{m} and whose edges are dual to those of \mathbf{m} . The origin of \mathbf{m}^\dagger is the root face f_r of \mathbf{m} .

Let \mathfrak{e}° be a finite connected subset of edges of \mathbf{m}^\dagger such that the origin of \mathbf{m}^\dagger is incident to \mathfrak{e}° (the letter “ \mathfrak{e} ” stands for *explored*). We associate with \mathfrak{e}° a planar map \mathfrak{e} which, roughly speaking, is obtained by gluing the faces of \mathbf{m} corresponding to the vertices incident to \mathfrak{e}° along the (dual) edges of \mathfrak{e}° , see Fig. 2. The resulting map, rooted at the root edge of \mathbf{m} , is a finite (rooted bipartite) planar map with several distinguished faces $h_1, \dots, h_k \in \text{Faces}(\mathfrak{e})$ that correspond to the connected components of $\mathbf{m}^\dagger \setminus \mathfrak{e}^\circ$. These distinguished faces are called the holes of \mathfrak{e} . Notice that the holes are simple, meaning that there is no pinch-point on their boundaries, and these boundaries also do not have vertices in common. Such an object will be called a planar map with holes. See Fig. 2 below.

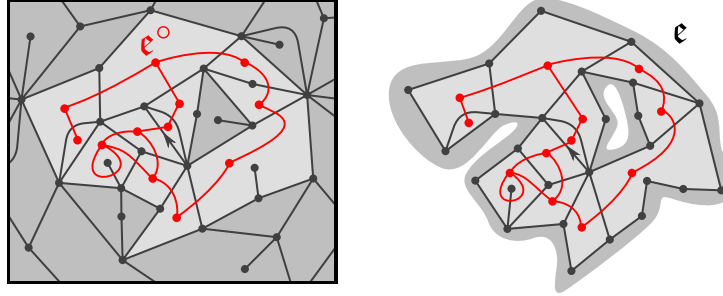


Figure 2: Illustration of the duality between connected subsets of edges on the dual map and their associated submaps on the primal lattice.

We say that \mathfrak{e} is a submap of \mathbf{m} and write $\mathfrak{e} \subset \mathbf{m}$ since \mathbf{m} can be obtained back from \mathfrak{e} by gluing inside each hole h_i of \mathfrak{e} a (uniquely defined) bipartite planar map \mathbf{u}_i of perimeter $\deg(h_i)$ (the letter \mathbf{u} stands for *unexplored*). To perform this gluing operation, we implicitly assume that an oriented edge is distinguished on the boundary of each hole h_i of \mathfrak{e} , on which we glue the root edge of \mathbf{u}_i . We will not mention this further, since these edges can be arbitrarily chosen using a deterministic procedure given \mathfrak{e} . Notice that after this gluing operation, it might happen that several edges on the boundary of a given hole of \mathfrak{e} get identified because the boundary of \mathbf{u}_i may not be simple, see Fig. 3 below. We will alternatively speak of “gluing” as “filling-in the hole”.

It is easy to see that this operation is rigid (see [2, Definition 4.7]) in the sense that if $\mathfrak{e} \subset \mathbf{m}$, then the maps $(\mathbf{u}_i)_{1 \leq i \leq k}$ are uniquely defined (in other words, if one glues different maps inside a given planar map with holes, one gets different maps after the gluing procedure). This definition even makes sense when \mathfrak{e} is a finite map and \mathbf{m} is an infinite map. Conversely, if $\mathfrak{e} \subset \mathbf{m}$, one can recover \mathfrak{e}° in a unique way

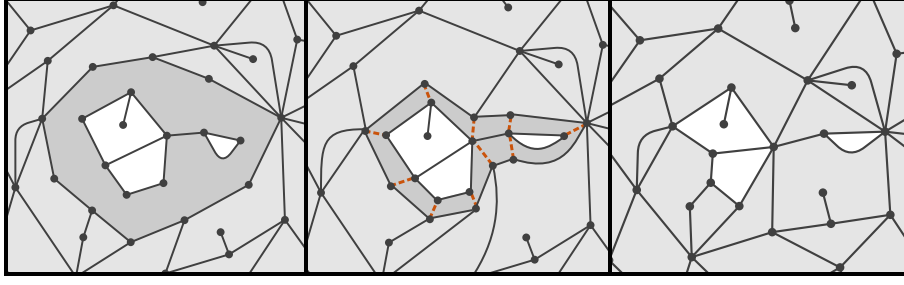


Figure 3: The operation of gluing a planar map into a hole.

as the set of all the dual edges between faces of \mathfrak{e} which are not holes.

This discussion shows that there are two points of view on submaps of \mathfrak{m} which are equivalent: either submaps can be seen as objects of the type of \mathfrak{e}° (which are connected components of edges containing the origin in \mathfrak{m}^\dagger), or as planar maps $\mathfrak{e} \subset \mathfrak{m}$ with holes (possibly none) which may be filled-in to obtain \mathfrak{m} . In this paper, we will mostly work with the second point of view.

6.2.2 Branching edge-peeling explorations

We now define the branching edge-peeling exploration, which is a means to explore a planar map edge after edge. If \mathfrak{e} is a planar map with holes, a cycle⁴ of \mathfrak{e} is by definition a connected subset of edges adjacent to a hole of \mathfrak{e} . We denote by $\mathcal{C}(\mathfrak{e})$ the union of the cycles of \mathfrak{e} . Formally, a branching peeling exploration depends on a function \mathcal{A} , called the *peeling algorithm*, which associates with any planar map with holes \mathfrak{e} an edge of $\mathcal{C}(\mathfrak{e}) \cup \{\dagger\}$, where \dagger is a cemetery point which we interpret as ending the exploration. In particular, if \mathfrak{e} has no holes, we must have $\mathcal{A}(\mathfrak{e}) = \dagger$. We say that this peeling algorithm is deterministic since no randomness is involved in the definition of \mathcal{A} .

Intuitively speaking, given the peeling algorithm \mathcal{A} , the branching edge-peeling process of a planar map \mathfrak{m} is a way to iteratively explore \mathfrak{m} starting from its boundary and discovering at each step a new edge by *peeling an edge* determined by the algorithm \mathcal{A} . If $\mathfrak{e} \subset \mathfrak{m}$ is a planar map with holes and e is an edge belonging to a cycle \mathcal{C} of \mathfrak{e} , the planar map with holes \mathfrak{e}_e obtained by peeling e is defined as follows. Let F_e be the face of \mathfrak{m} that is adjacent to the same side of e as the hole in \mathfrak{e} . Then there are two possibilities, see Fig. 4:

- *Event C_k* : the face F_e is not a face of \mathfrak{e} and has degree $2k$. Then \mathfrak{e}_e is obtained by gluing F_e on e .
- *Event G_{k_1, k_2}* : the face F_e is actually a face of \mathfrak{e} . In this case, the edge e is identified in \mathfrak{m} with another edge e' of the same cycle \mathcal{C} where $2k_1$ (resp. $2k_2$) is the number of edges of \mathcal{C} strictly between e and e' when turning in clockwise order around the cycle, and \mathfrak{e}_e is the map after this identification in \mathfrak{e} .

When $k_1 > 0$ and $k_2 > 0$, note that the event G_{k_1, k_2} results in the splitting of a hole into two holes, and the event $G_{0,0}$ results in the disappearance of a hole.

Formally, if \mathfrak{m} is a (finite or infinite) planar map, the branching edge-peeling exploration of \mathfrak{m} with algorithm \mathcal{A} is by definition the sequence of planar maps with holes

$$\mathfrak{e}_0(\mathfrak{m}) \subset \mathfrak{e}_1(\mathfrak{m}) \subset \cdots \subset \mathfrak{e}_n(\mathfrak{m}) \subset \cdots \subset \mathfrak{m},$$

⁴Contrary to [6], the cycles cannot be seen as self-avoiding loops on the original map \mathfrak{m} since they are closed paths which may visit twice the same edge; they are called frontiers in [15].

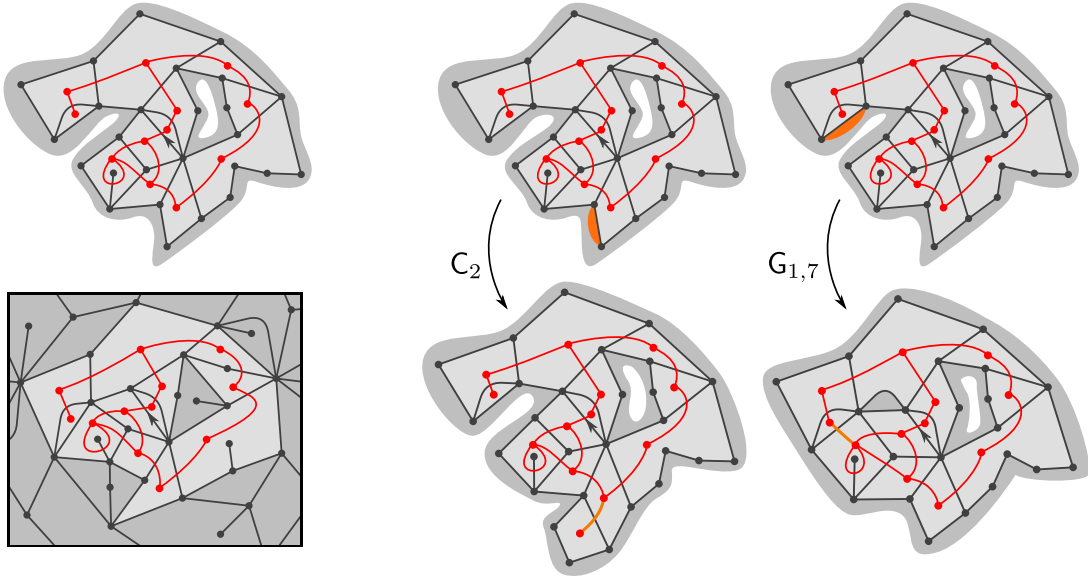


Figure 4: Illustration of the different edge-peeling events. The left column illustrates the submap $\mathfrak{e} \subset \mathfrak{m}$. The center and right columns represent two different peeling events, with the edge to be peeled indicated in orange.

obtained as follows:

- the map $\mathfrak{e}_0(\mathfrak{m})$ is made of a simple face corresponding to the root face f_r of the map and a unique hole of the same perimeter;
- for every $i \geq 0$, if $\mathcal{A}(\mathfrak{e}_i(\mathfrak{m})) \neq \dagger$, then the planar map with holes $\mathfrak{e}_{i+1}(\mathfrak{m})$ is obtained from $\mathfrak{e}_i(\mathfrak{m})$ by peeling the edge $\mathcal{A}(\mathfrak{e}_i(\mathfrak{m}))$. If $\mathcal{A}(\mathfrak{e}_i(\mathfrak{m})) = \dagger$, then $\mathfrak{e}_{i+1}(\mathfrak{m}) = \mathfrak{e}_i(\mathfrak{m})$ and the exploration process stops.

In particular, observe that if $\mathfrak{e}_i(\mathfrak{m}) \neq \mathfrak{e}_{i-1}(\mathfrak{m})$ with $i \geq 1$, then $\mathfrak{e}_i(\mathfrak{m})$ has exactly i internal edges (which are by definition edges of $\mathfrak{e}_i(\mathfrak{m})$ which do not belong to cycles). If $i \geq 0$, the map with holes $\mathfrak{e}_i(\mathfrak{m})$ is obviously a (deterministic) function of \mathfrak{m} . But note that $(\mathfrak{e}_j(\mathfrak{m}); 0 \leq j \leq i)$ is also a (deterministic) function of $\mathfrak{e}_i(\mathfrak{m})$. Finally, to simplify notation, we will often write \mathfrak{e}_i instead of $\mathfrak{e}_i(\mathfrak{m})$.

Remark 6.2. At this point, the reader may compare the above presentation with that of [6, Sec. 2.3]. In the peeling process considered in [6, Sec. 2.3], the sequence $\mathfrak{e}_0 \subset \dots \subset \mathfrak{e}_n \subset \dots \subset \mathfrak{m}$ is again a sequence of maps with simple holes (with the slight difference that in this case the holes can share vertices but not edges) but (unless the peeling has stopped), \mathfrak{e}_{i+1} is obtained from \mathfrak{e}_i by the addition of a new *face*. Furthermore, in this peeling process, \mathfrak{m} is obtained from \mathfrak{e}_i by the filling-in the holes of \mathfrak{e}_i with maps having *simple* boundary. In other words, the peeling process of [6, Sec. 2.3] is “face”-peeling, while in the present work we have an “edge”-peeling.

Remark 6.3. One can alternatively represent a peeling exploration $\mathfrak{e}_0 \subset \mathfrak{e}_1 \subset \dots \subset \mathfrak{e}_n \subset \dots \subset \mathfrak{m}$ as the associated sequence of growing connected subsets of edges $(\mathfrak{e}_i^\circ)_{i \geq 0}$ of the dual map \mathfrak{m}^\dagger , such that \mathfrak{e}_{i+1}° is obtained from \mathfrak{e}_i° by adding one edge of \mathfrak{m}^\dagger (unless the exploration has stopped). We will however mostly use the first point of view.

In the sequel, we usually simply say peeling exploration instead of branching edge-peeling exploration. Notice also that in our notation the sequence of explored maps (\mathbf{e}_i) depends obviously on the underlying map, but also on the peeling algorithm \mathcal{A} . In the following it should be clear from the context which statements are valid for all peeling explorations and which for specific ones.

6.3 Peeling of random Boltzmann maps

When dealing with finite or infinite Boltzmann planar maps, we work on the canonical space Ω of all the (rooted bipartite, possibly infinite) random maps with holes, possibly pointed. This space is equipped with the Borel σ -field for the local topology. The notation

$$\mathbb{P}^{(\ell)}, \mathbb{E}^{(\ell)}, \quad \text{resp. } \mathbb{P}_{\infty}^{(\ell)}, \mathbb{E}_{\infty}^{(\ell)}, \quad \text{resp. } \mathbb{P}_{\bullet}^{(\ell)}, \mathbb{E}_{\bullet}^{(\ell)}$$

are used for the probability and expectation on Ω relative to the law of a \mathbf{q} -Boltzmann map with perimeter 2ℓ , resp. the infinite \mathbf{q} -Boltzmann map with perimeter 2ℓ , resp. a pointed \mathbf{q} -Boltzmann map with perimeter 2ℓ . A generic element of the canonical space will be either denoted by \mathbf{m} or by $\mathbf{m}_{\bullet} = (\mathbf{m}, v_{\bullet})$ if it is pointed.

In this section, we fix a peeling algorithm \mathcal{A} and first compute the law of the branching edge-peeling exploration $\mathbf{e}_0 \subset \mathbf{e}_1 \subset \dots \subset \mathbf{m}$ under $\mathbb{P}^{(\ell)}$, $\mathbb{P}_{\bullet}^{(\ell)}$ and $\mathbb{P}_{\infty}^{(\ell)}$. We denote by \mathcal{F}_n the σ -algebra on Ω generated by the functions $\mathbf{m} \mapsto \mathbf{e}_0(\mathbf{m}), \mathbf{m} \mapsto \mathbf{e}_1(\mathbf{m}), \dots, \mathbf{m} \mapsto \mathbf{e}_n(\mathbf{m})$. If \mathbf{e} is a planar map with holes, we set

$$\tilde{\mathbf{w}}(\mathbf{e}) = \prod_{\substack{f \in \text{Faces}(\mathbf{e}) \setminus \{f_{\tau}\} \\ f \text{ is not a hole}}} q_{\deg(f)/2}$$

and let $|\mathbf{e}|$ be the number of internal vertices of \mathbf{e} (an internal vertex of \mathbf{e} is a vertex that does not belong to a cycle of \mathbf{e}). Finally, to simplify notation, for every $\ell \geq 0$, we set

$$f^{\uparrow}(\ell) = \frac{h^{\uparrow}(\ell)}{W^{(\ell)}\gamma^{\ell}} \quad \text{and} \quad f^{\curvearrowright}(\ell) = \frac{h^{\curvearrowright}(\ell)}{W^{(\ell)}\gamma^{\ell}} \stackrel{(34)}{=} \frac{W_{\bullet}^{(\ell)}}{W^{(\ell)}}.$$

Note that, by (33) we have for some $c, c' > 0$

$$f^{\uparrow}(\ell) \underset{\ell \rightarrow \infty}{\sim} c \cdot \ell^{\theta+3/2}, \quad f^{\curvearrowright}(\ell) \underset{\ell \rightarrow \infty}{\sim} c' \cdot \ell^{\theta+1/2}. \quad (38)$$

Proposition 6.4. *Fix $\ell \geq 1$ and $n \geq 0$. Let \mathbf{e} be a planar map with holes which can be obtained after n peeling steps starting from a simple face of perimeter 2ℓ . Denote by $\ell_1, \ell_2, \dots, \ell_k$ the half-perimeters of the holes of \mathbf{e} . Then*

$$\mathbb{P}^{(\ell)}(\mathbf{e}_n = \mathbf{e}) = \frac{\tilde{\mathbf{w}}(\mathbf{e})}{W^{(\ell)}} \prod_{i=1}^k W^{(\ell_i)}, \quad \mathbb{P}_{\bullet}^{(\ell)}(\mathbf{e}_n = \mathbf{e}) = \frac{1}{f^{\curvearrowright}(\ell)} \left(|\mathbf{e}| + \sum_{j=1}^k f^{\curvearrowright}(\ell_j) \right) \cdot \mathbb{P}^{(\ell)}(\mathbf{e}_n = \mathbf{e}) \quad (39)$$

and

$$\mathbb{P}_{\infty}^{(\ell)}(\mathbf{e}_n = \mathbf{e}) = \frac{1}{f^{\uparrow}(\ell)} \left(\sum_{j=1}^k f^{\uparrow}(\ell_j) \right) \cdot \mathbb{P}^{(\ell)}(\mathbf{e}_n = \mathbf{e}). \quad (40)$$

Furthermore:

- (i) Under $\mathbb{P}^{(\ell)}$ and conditionally on $\{\mathbf{e}_n = \mathbf{e}\}$, the random maps filling-in the holes of \mathbf{e}_n inside \mathbf{m} are independent \mathbf{q} -Boltzmann maps.

- (ii) Under $\mathbb{P}_{\bullet}^{(\ell)}$, conditionally given that $\{\mathbf{e}_n = \mathbf{e}\}$ and that v_{\bullet} is not an internal vertex of \mathbf{e} , the planar maps filling-in the holes of \mathbf{e}_n inside \mathbf{m} are independent, all being \mathbf{q} -Boltzmann maps, except for the J -th hole which is filled-in with a pointed \mathbf{q} -Boltzmann map distributed as $\mathbb{P}_{\bullet}^{(\ell_J)}$, where the index J is chosen at random, independently and proportionally to $j \mapsto f^{\vee}(\ell_j)$.
- (iii) Under $\mathbb{P}_{\infty}^{(\ell)}$ and conditionally on $\{\mathbf{e}_n = \mathbf{e}\}$, the planar maps filling-in the holes of \mathbf{e}_n inside \mathbf{m} are independent, all being \mathbf{q} -Boltzmann maps, except for the J -th hole which is filled-in with an infinite \mathbf{q} -Boltzmann map of perimeter $2\ell_J$ where the index J is chosen at random, independently and proportionally to $j \mapsto f^{\uparrow}(\ell_j)$.

Proof. Since the peeling algorithm is deterministic and by rigidity, the event $\{\mathbf{e}_n = \mathbf{e}\}$ happens if and only if \mathbf{m} is obtained from \mathbf{e} by filling-in the holes of \mathbf{e} with certain maps. The assertions concerning $\mathbb{P}^{(\ell)}$ simply follow from the definition of the Boltzmann measure. The identities concerning $\mathbb{P}_{\infty}^{(\ell)}$ are obtained in a similar way to those of Proposition 3 in [6], but we give a full proof for sake of completeness. We have

$$\begin{aligned} \mathbb{P}^{(\ell)}(\mathbf{e}_n(\mathbf{m}) = \mathbf{e} \mid |\mathbf{m}| = N) &= \frac{\tilde{\mathbf{w}}(\mathbf{e})}{W_N^{(\ell)}} \sum_{\substack{n_1 + \dots + n_k = N - |\mathbf{e}| \\ \mathbf{m}_1 \in \text{Map}_{n_1}^{(\ell_1)}, \dots, \mathbf{m}_k \in \text{Map}_{n_k}^{(\ell_k)}}} \mathbf{w}(\mathbf{m}_1) \cdots \mathbf{w}(\mathbf{m}_k) \\ &= \frac{\tilde{\mathbf{w}}(\mathbf{e})}{W_N^{(\ell)}} \sum_{n_1 + \dots + n_k = N - |\mathbf{e}|} W_{n_1}^{(\ell_1)} \cdots W_{n_k}^{(\ell_k)}. \end{aligned}$$

Using Proposition 6.1, it is then an easy matter to verify that, for any $\varepsilon > 0$, we can choose K sufficiently large so that as $N \rightarrow \infty$, the asymptotic contribution of terms corresponding to choices of n_1, \dots, n_k where $n_i \geq K$ for two distinct values of $i \in \{1, \dots, m\}$ is bounded above by ε (see [2, Lemma 2.5]), so that by Proposition 6.1

$$\mathbb{P}^{(\ell)}(\mathbf{e}_n(\mathbf{m}) = \mathbf{e} \mid |\mathbf{m}| = N) \xrightarrow{N \rightarrow \infty} \frac{\tilde{\mathbf{w}}(\mathbf{e})}{\gamma^{-\ell} h^{\uparrow}(\ell)} \cdot \sum_{i=1}^k \gamma^{-\ell_i} h^{\uparrow}(\ell_i) \prod_{\substack{j=1 \\ j \neq i}}^k W^{(\ell_j)} = \frac{1}{f^{\uparrow}(\ell)} \left(\sum_{j=1}^k f^{\uparrow}(\ell_j) \right) \cdot \mathbb{P}^{(\ell)}(\mathbf{e}_n = \mathbf{e}).$$

By [35, Theorem 6.1], $\mathbb{P}^{(\ell)}(\mathbf{e}_n(\mathbf{m}) = \mathbf{e} \mid |\mathbf{m}| = N) \rightarrow \mathbb{P}_{\infty}^{(\ell)}(\mathbf{e}_n = \mathbf{e})$. This completes the proof of (40). The assertion (iii) is established by using similar arguments and is left to the reader.

Let us now prove the assertions concerning $\mathbb{P}_{\bullet}^{(\ell)}$. By definition of $\mathbb{P}_{\bullet}^{(\ell)}$ (see (32)),

$$\begin{aligned} \mathbb{P}_{\bullet}^{(\ell)}(\mathbf{e}_n = \mathbf{e}, v_{\bullet} \text{ is an internal vertex of } \mathbf{e}) &= \frac{|e| \tilde{\mathbf{w}}(\mathbf{e})}{W_{\bullet}^{(\ell)}} \cdot \sum_{\mathbf{m}_1 \in \text{Map}^{(\ell_1)}, \dots, \mathbf{m}_k \in \text{Map}^{(\ell_k)}} \mathbf{w}(\mathbf{m}_1) \cdots \mathbf{w}(\mathbf{m}_k) \\ &= \frac{|e| \tilde{\mathbf{w}}(\mathbf{e})}{W_{\bullet}^{(\ell)}} \cdot W^{(\ell_1)} \cdots W^{(\ell_k)} = \frac{\tilde{\mathbf{w}}(\mathbf{e})}{W^{(\ell)}} \left(\prod_{i=1}^k W^{(\ell_i)} \right) \cdot \frac{1}{f^{\vee}(\ell)} \cdot |e|, \end{aligned}$$

where we have used the fact that $f^{\vee}(\ell) = W_{\bullet}^{(\ell)} / W^{(\ell)}$ for the last equality. Similarly, for $1 \leq j \leq k$,

$$\begin{aligned} \mathbb{P}_{\bullet}^{(\ell)}(\mathbf{e}_n = \mathbf{e}, v_{\bullet} \in \mathbf{m}_j) &= \frac{\tilde{\mathbf{w}}(\mathbf{e})}{W_{\bullet}^{(\ell)}} \cdot \sum_{\mathbf{m}_1 \in \text{Map}^{(\ell_1)}, \dots, \mathbf{m}_k \in \text{Map}^{(\ell_k)}} |\mathbf{m}_j| \cdot \mathbf{w}(\mathbf{m}_1) \cdots \mathbf{w}(\mathbf{m}_k) \\ &= \frac{\tilde{\mathbf{w}}(\mathbf{e})}{W_{\bullet}^{(\ell)}} \cdot W^{(\ell_1)} \cdots W^{(\ell_k)} \cdot \frac{W_{\bullet}^{(\ell_j)}}{W^{(\ell_j)}} = \frac{\tilde{\mathbf{w}}(\mathbf{e})}{W^{(\ell)}} \left(\prod_{i=1}^k W^{(\ell_i)} \right) \cdot \frac{1}{f^{\vee}(\ell)} \cdot f^{\vee}(\ell_j). \end{aligned}$$

The expression of $\mathbb{P}_{\bullet}^{(\ell)}(\mathbf{e}_n = \mathbf{e})$ then follows by summing the previous equalities. Assertion (ii) is established in a similar way, and we leave details to the reader. \square

We now exhibit two martingales that appear in every peeling exploration of a \mathbf{q} -Boltzmann planar map. As before, we denote by $\mathfrak{e}_0 \subset \dots \subset \mathfrak{e}_n \subset \dots$ the branching peeling exploration of an underlying planar map \mathfrak{m} (recall that the peeling algorithm \mathcal{A} is fixed). To simplify notation, for every $n \geq 0$, we let $\ell(n) = (\ell_1(n), \ell_2(n), \dots)$ be the lengths of the holes of the explored map \mathfrak{e}_n . If g is a function and $\ell = (\ell_1, \ell_2, \dots)$ is a sequence of integers we simply write $g(\ell)$ for the sum $g(\ell_1) + g(\ell_2) + \dots$.

Proposition 6.5. *Under $\mathbb{P}^{(\ell)}$, the processes*

$$\left(f^\uparrow(\ell(n))\right)_{n \geq 0} \quad \text{and} \quad \left(|\mathfrak{e}_n| + f^\vee(\ell(n))\right)_{n \geq 0}$$

are positive (\mathcal{F}_n) martingales, which are respectively called the cycle and the area martingales. The cycle martingale is not uniformly integrable and converges almost surely to 0, whereas the area martingale is closed and at time n is equal to the conditional expectation of the total number of vertices of a \mathbf{q} -Boltzmann planar map given the map with holes obtained after n peeling steps.

Remark 6.6. With the connection with growth-fragmentation processes that we have in mind, the above two martingales are the discrete analogs of respectively the martingale M^+ and the martingale M^- studied in the preceding sections.

Proof. The proofs are the same as that of Proposition 6 and 7 in [6]. More precisely, from Proposition 6.4 it follows that $f^\uparrow(\ell(n))/f^\uparrow(\ell)$ is the Radon-Nikodym derivative of the branching peeling exploration until step n under $\mathbb{P}_\infty^{(\ell)}$ with respect to the same exploration under $\mathbb{P}^{(\ell)}$. This implies the first assertion. For the second one, we similarly rely on Proposition 6.4 and observe that

$$\mathbb{E}^{(\ell)}(|\mathfrak{m}|) = \frac{W_\bullet^{(\ell)}}{W^{(\ell)}} \stackrel{(34)}{=} f^\vee(\ell). \quad (41)$$

□

6.4 Scaling limits for the perimeter of a distinguished cycle

In this section, we establish a scaling limit result for the perimeter of a distinguished cycle under $\mathbb{P}^{(\ell)}$, $\mathbb{P}_\bullet^{(\ell)}$ and $\mathbb{P}_\infty^{(\ell)}$. We fix a peeling algorithm \mathcal{A} such that $\mathcal{A}(\mathfrak{m}) \neq \dagger$ if \mathfrak{m} has at least one hole. Recall that $(S_n)_{n \geq 0}$ denotes the random walk on \mathbb{Z} with jump distribution ν .

Infinite Boltzmann planar maps. When performing a branching peeling exploration under $\mathbb{P}_\infty^{(\ell)}$, there is one cycle that plays a particular role, namely the one separating the origin from infinity (the infinite version of a \mathbf{q} -Boltzmann planar map has almost surely one end [35, Lemma 6.3]). Specifically, during a peeling exploration of an infinite planar map with one end, we define a family of distinguished cycles $(\mathcal{C}_\infty(i))_{i \geq 0}$ as follows. The initial distinguished cycle $\mathcal{C}_\infty(0)$ is the only cycle of \mathfrak{e}_0 and $\sigma_0 = 0$. Then, inductively, for $i \geq 0$, if $\mathcal{C}_\infty(i) = \dagger$ (the cemetery point), set $\mathcal{C}_\infty(i+1) = \dagger$, and otherwise define $\sigma_{i+1} = \inf\{j > \sigma_i : \mathcal{A}(\mathfrak{e}_j) \in \mathcal{C}_\infty(i)\}$ (with the usual convention $\inf \emptyset = \infty$). If $\sigma_{i+1} = \infty$, we define $\mathcal{C}_\infty(i+1) = \mathcal{C}_\infty(i)$. Otherwise, when peeling the edge $\mathcal{A}(\mathfrak{e}_{\sigma_{i+1}})$, we define $\mathcal{C}_\infty(i+1)$ depending on what peeling event happens:

- If the event C_k occurs, we define $\mathcal{C}_\infty(i+1)$ to be the new cycle thus created,
- If the event $G_{0,0}$ occurs (disappearance of the hole), we define $\mathcal{C}_\infty(i+1) = \dagger$,
- If the event G_{k_1, k_2} occurs with $(k_1, k_2) \neq (0, 0)$, two new cycles are created (one possibly empty) when peeling the edge $\mathcal{A}(\mathfrak{e}_{\sigma_{i+1}})$. We define $\mathcal{C}_\infty(i+1)$ to be the cycle that separates the origin of the map from infinity.

Observe that under $\mathbb{P}_\infty^{(\ell)}$, the event $\mathbf{G}_{0,0}$ has probability 0. Finally, we agree by convention that the perimeter of \dagger is 0, and for every $i \geq 0$ we let $P_\infty(i)$ be the half-perimeter of $\mathcal{C}_\infty(i)$.

It follows from Proposition 6.4 that, under $\mathbb{P}_\infty^{(\ell)}$, the process $(P_\infty(i) : i \geq 0)$ evolves as a Markov chain on the positive integers starting from ℓ with explicit transitions:

$$\mathbb{P}_\infty^{(\ell)}(P_\infty(i+1) = k+m \mid P_\infty(i) = m) = \nu(k) \cdot \frac{h^\uparrow(m+k)}{h^\uparrow(m)} \quad (k \geq -m).$$

This Markov chain can be seen as the random walk $(S_n)_{n \geq 0}$ starting from ℓ conditioned to remain positive, that is, rigorously, as the Doob h^\uparrow -transform of the random walk $(S_n)_{n \geq 0}$ killed when entering \mathbb{Z}_- (see [15], [16, Sec. 1.2.2]).

Pointed Boltzmann planar maps. When performing a branching peeling exploration under $\mathbb{P}_\bullet^{(\ell)}$, there is one cycle that plays a particular role, namely the one separating the origin from v_\bullet . Specifically, we define a family of distinguished cycles $(\mathcal{C}_\bullet(i))_{i \geq 0}$ exactly as in the case of infinite Boltzmann planar maps, with the only difference that if the event \mathbf{G}_{k_1, k_2} occurs we define $\mathcal{C}_\bullet(i+1)$ to be the cycle which is filled-in by the planar map containing the distinguished vertex v_\bullet . If the latter cycle is empty, i.e. when v_\bullet is encountered, we set $\mathcal{C}_\bullet(i+1) = \dagger$. Finally, for every $i \geq 0$ we let $P_\bullet(i)$ be the half-perimeter of $\mathcal{C}_\bullet(i)$.

It follows from Proposition 6.4 that, under $\mathbb{P}_\bullet^{(\ell)}$, the process $(P_\bullet(i) : i \geq 0)$ evolves as a Markov chain on the non-negative integers starting from ℓ with the following explicit transitions:

$$\mathbb{P}_\bullet^{(\ell)}(P_\bullet(i+1) = k+m \mid P_\bullet(i) = m) = \nu(k) \cdot \frac{h^\vee(m+k)}{h^\vee(m)} \quad (k \geq -m).$$

This Markov chain can therefore be seen as the random walk $(S_n)_{n \geq 0}$ starting from ℓ , absorbed when entering \mathbb{Z}_- and conditioned to be absorbed at 0, or, equivalently, as a Doob h^\vee -transform of the random walk $(S_n)_{n \geq 0}$ killed when entering \mathbb{Z}_- .

Boltzmann planar maps. During a peeling exploration under $\mathbb{P}^{(\ell)}$, we can still distinguish a natural family of cycles, namely the locally largest ones. Specifically, we define a family of distinguished cycles $(\mathcal{C}_*(i))_{i \geq 0}$ exactly as in the case of infinite Boltzmann planar maps, with the only difference that if the event \mathbf{G}_{k_1, k_2} occurs with $(k_1, k_2) \neq (0, 0)$, one creates two new cycles (one possibly empty), and we define $\mathcal{C}_*(i+1)$ to be the cycle with largest perimeter (if $\ell_1 = \ell_2$, we choose between the two in a deterministic way). Finally, for every $i \geq 0$ we let $P_*(i)$ be the half-perimeter of $\mathcal{C}_*(i)$.

Again, Proposition 6.4 shows that under $\mathbb{P}^{(\ell)}$, $(P_*(i) : i \geq 0)$ is a Markov chain on the positive integers starting from ℓ with the following explicit transitions:

$$\mathbb{P}^{(\ell)}(P_*(i+1) = k+m \mid P_*(i) = m) = \nu(k) \cdot \frac{\gamma^{m+k} W^{(m+k)}}{\gamma^m W^{(m)}} \quad (k > \frac{-m-1}{2}),$$

and the transition is just half of the last display if $m-1$ is even and $-k = \frac{m+1}{2}$. However, we will not need the exact value of these transitions in the following.

Scaling limits for the perimeter of the distinguished cycle. We now introduce the different scaling limits of the three families of distinguished cycles we have just defined. They are all related to the growth-fragmentation $(\mathbf{X}_\theta^{(-\theta)}(t), t \geq 0)$ with cumulant function κ_θ given by (19) and self-similarity parameter $-\theta$.

Using the results of Sec. 5.2; we may construct $\mathbf{X}_\theta^{(-\theta)}$ by choosing the evolution of the Eve cell to be the self-similar Markov process $X_\theta^{(-\theta)}$ with characteristics $(\Psi_\theta, -\theta)$, where we recall that Ψ_θ is the Laplace exponent (28). In particular, $X_\theta^{(-\theta)}$ does not make negative jumps larger than half of its current value and, roughly speaking, describes the evolution of the size of the locally largest particle. Recalling the results of the last section, the processes Y_θ^+ and Y_θ^- defined in Sec. 4.1 and Sec. 4.3 corresponding to the evolution of the tagged particles in the biased versions of $\mathbf{X}_\theta^{(-\theta)}$ are distributed as follows: Let Υ_θ be the θ -stable Lévy process with positivity parameter $\rho = \mathbb{P}(\Upsilon_\theta(1) \geq 0)$ satisfying $\theta(1 - \rho) = 1/2$ and normalized so that its Lévy measure is

$$\frac{\Gamma(1 + \theta)}{\pi} \cdot \cos((1 + \theta)\pi) \cdot \frac{dx}{x^{1+\theta}} \mathbb{1}_{x>0} + \frac{\Gamma(1 + \theta)}{\pi} \cdot \frac{dx}{|x|^{1+\theta}} \mathbb{1}_{x<0}.$$

Then introduce Υ_θ^\uparrow and $\Upsilon_\theta^\downarrow$ the versions of the Lévy process Υ_θ conditioned to stay positive and respectively to die continuously at 0 when it enters \mathbb{R}_- (see [18, 17] for the definition of these processes). Then by the result of the last section we have $Y_\theta^+ = \Upsilon_\theta^\uparrow$ and $Y_\theta^- = \Upsilon_\theta^\downarrow$ in law.

Proposition 6.7. *Assume that \mathbf{q} is admissible, critical, non-generic and satisfies $q_k \sim c \cdot \gamma^{k-1} \cdot k^{-1-\theta}$ as $k \rightarrow \infty$, with $\theta \in (1/2, 1) \cup (1, 3/2)$. Then, setting $\mathbf{c}_\mathbf{q} = \frac{\pi c}{\Gamma(1+\theta) \cos((1+\theta)\pi)}$ the following three convergences hold in distribution for the Skorokhod J_1 topology on $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$:*

$$\begin{aligned} \text{under } \mathbb{P}_\infty^{(\ell)}, \quad & \left(\frac{1}{\ell} \cdot P_\infty(\lfloor \ell^\theta \cdot t \rfloor) : t \geq 0 \right) \xrightarrow[\ell \rightarrow \infty]{(d)} (Y_\theta^+(\mathbf{c}_\mathbf{q} \cdot t) : t \geq 0); \\ \text{under } \mathbb{P}^{(\ell)}, \quad & \left(\frac{1}{\ell} \cdot P_*(\lfloor \ell^\theta \cdot t \rfloor) : t \geq 0 \right) \xrightarrow[\ell \rightarrow \infty]{(d)} (X_\theta^{(-\theta)}(\mathbf{c}_\mathbf{q} \cdot t) : t \geq 0); \\ \text{under } \mathbb{P}_\bullet^{(\ell)}, \quad & \left(\frac{1}{\ell} \cdot P_\bullet(\lfloor \ell^\theta \cdot t \rfloor) : t \geq 0 \right) \xrightarrow[\ell \rightarrow \infty]{(d)} (Y_\theta^-(\mathbf{c}_\mathbf{q} \cdot t) : t \geq 0). \end{aligned}$$

Proof. The first assertion is [16, Proposition 3], but with another constant due to the different normalization of the Lévy measure. The constant $\mathbf{c}_\mathbf{q}$ can be easily computed by using the tail of the ν :

$$\nu([k, \infty)) \underset{k \rightarrow \infty}{\sim} \frac{c}{\theta} \cdot \frac{1}{k^\theta} \quad \text{and} \quad \nu((-\infty, -k]) \underset{k \rightarrow \infty}{\sim} \frac{c}{\theta \cos((1 + \theta)\pi)} \cdot \frac{1}{k^\theta}.$$

For the third assertion, we have seen that under $\mathbb{P}_\bullet^{(\ell)}$, the process $(P_\bullet(i) : i \geq 0)$ evolves as the random walk $(W_n)_{n \geq 0}$ starting from ℓ , absorbed when entering \mathbb{Z}_- and conditioned to be absorbed at 0, or, equivalently as the Doob h^\downarrow -transform of the random walk $(W_n)_{n \geq 0}$ absorbed when entering \mathbb{Z}_- . The third assertion then follows from an invariance principle of Caravenna & Chaumont [17, Theorem 1.3].

For the second assertion, instead of adapting the proof of [6, Proposition 9] and applying results of [8], we shall transfer a convergence in distribution under $\mathbb{P}_\infty^{(\ell)}$ to a convergence under $\mathbb{P}^{(\ell)}$ by using absolute continuity relations in both the discrete setting (relying on Sec. 6.3) and the continuous setting (relying on Prop. 4.1). Specifically, Proposition 6.4 shows that for every $n \geq 0$ and every sequence $x_0 = \ell, x_1, \dots, x_n$ in $\{1, 2, \dots\}$ with $x_{i+1} \geq \frac{1}{2}x_i$ for all $i = 0, \dots, n-1$, there is the identity

$$\mathbb{P}^{(\ell)}(P_*(0) = x_0, \dots, P_*(n) = x_n) = \frac{f^\uparrow(\ell)}{f^\uparrow(x_n)} \mathbb{P}_\infty^{(\ell)}(P_\infty(0) = x_0, \dots, P_\infty(n) = x_n).$$

As a consequence, if we fix $T > 0$ and let $F : \mathbb{D}([0, T], \mathbb{R}) \rightarrow \mathbb{R}_+$ be a bounded continuous function such that $F(X) = 0$ if $X \in \mathbb{D}([0, T], \mathbb{R})$ is such that there exists $0 < t \leq T$ with $X(t) < X(t-)/2$, we get that

$$\mathbb{E}^{(\ell)} \left(F \left(\frac{1}{\ell} P_*(\lfloor \ell^\theta t \rfloor) : 0 \leq t \leq T \right) \right) \xrightarrow[\ell \rightarrow \infty]{} \mathbb{E} \left(Y_\theta^+(\mathbf{c}_\mathbf{q} \cdot T)^{-(\theta+3/2)} F(Y_\theta^+(\mathbf{c}_\mathbf{q} \cdot t) : 0 \leq t \leq T) \right),$$

where we have also used the estimate (38) concerning the asymptotic behavior of $f^\uparrow(\ell)$.

Assume that the growth-fragmentation $(\mathbf{X}_\theta^{(-\theta)}(t), t \geq 0)$ is constructed by using the self-similar Markov process $X_\theta^{(-\theta)}$ for the evolution of the Eve cell. Recall from Sec. 4 the notation $\widehat{\mathcal{P}}_1^+$ describing the joint distribution of a cell system and a leaf, and that $\hat{\mathcal{X}}(t)$ is the size of the tagged cell at time t . In particular, we have $\omega_+ = \theta + 3/2$ (recall (20)). In addition, by Theorem 4.2, $\hat{\mathcal{X}}$ under $\widehat{\mathcal{P}}_1^+$ has the same distribution as Y_θ^+ . Therefore, setting $\mathbf{X}_\theta^{(-\theta)}(t) = \{X_1(t), X_2(t), \dots\}$,

$$\begin{aligned} \mathbb{E} \left(Y_\theta^+(T)^{-(\theta+3/2)} F(Y_\theta^+(t) : 0 \leq t \leq T) \right) &= \widehat{\mathcal{E}}_1^+ \left(\hat{\mathcal{X}}(T)^{-\omega_+} F(\hat{\mathcal{X}}(t) : 0 \leq t \leq T) \right) \\ &\stackrel{(\text{Prop. 4.1})}{=} \mathbb{E}_1 \left(\sum_{i=1}^{\infty} F(X_i(t) : 0 \leq t \leq T) \right) \\ &= E \left(F(X_\theta^{(-\theta)}(t) : 0 \leq t \leq T) \right). \end{aligned}$$

For the last equality, we have used the fact that if $X_i(T) > 0$ and $X_i(t) \geq X_i(t-)/2$ for every $0 < t < T$, then X_i is the Eve cell (it is the locally largest particle) and $X_i(t) = X_\theta^{(-\theta)}(t)$ for every $0 \leq t \leq T$. We conclude that

$$\mathbb{E}^{(\ell)} \left(F \left(\frac{1}{\ell} P_*([\ell^\theta t]) : 0 \leq t \leq T \right) \right) \xrightarrow{\ell \rightarrow \infty} \mathbb{E} \left(F(X_\theta^{(-\theta)}(\mathbf{c}_\mathbf{q} \cdot t) : 0 \leq t \leq T) \right).$$

This shows the second convergence and completes the proof. \square

Remark 6.8. The proposition above should remain true in the case $\theta = 1$. This case has been excluded for technical reasons in [16, Proposition 3] and we shall do the same here. However, it is known [16, Remark 3] that [16, Proposition 3] holds true for all $\theta \in (1/2, 3/2)$ in the case of the explicit weight sequence (31). Consequently the above proposition is valid for all $\theta \in (1/2, 3/2)$ for this sequence as well. We shall use this in the proof of Corollary 6.9.

As an application of Proposition 6.7, we compute the law of the intrinsic area of the growth-fragmentation $\mathbf{X}_\theta^{(-\theta)}$ (recall from Remark 3.11 that this law only depends on the cumulant function κ_θ). For $\beta \in (0, 1)$, let $\mathfrak{S}_\bullet(\beta)$ be a positive β -stable random variable with Laplace transform $\mathbb{E}(e^{-\lambda \mathfrak{S}_\bullet(\beta)}) = \exp(-(\Gamma(1 + 1/\beta)\lambda)^\beta)$. Then $\mathbb{E}(1/\mathfrak{S}_\bullet(\beta)) = 1$ and we can define a random variable $\mathfrak{S}(\beta)$ by biasing $\mathfrak{S}_\bullet(\beta)$ by $x \rightarrow 1/x$, that is for any $f \geq 0$

$$\mathbb{E}(f(\mathfrak{S}(\beta))) = \mathbb{E} \left(f(\mathfrak{S}_\bullet(\beta)) \frac{1}{\mathfrak{S}_\bullet(\beta)} \right).$$

Corollary 6.9. Fix $\theta \in (1/2, 3/2]$. Denote by $\mathcal{M}_\theta^-(\infty)$ the intrinsic area of a growth-fragmentation with cumulant function κ_θ . Then

$$\mathcal{M}_\theta^-(\infty) \stackrel{(d)}{=} \mathfrak{S} \left(\frac{1}{\theta + \frac{1}{2}} \right).$$

Proof. Fix $\theta \in (1/2, 3/2)$ and let \mathbf{q} be the admissible, critical, non-generic sequence given by (31) with polynomial tail of exponent $-\theta - 1$ so that the last proposition holds true even in the case $\theta = 1$. To simplify notation, we write \mathfrak{S} for the random variable $\mathfrak{S}((\theta + 1/2)^{-1})$ and we let $B^{(\ell)}$ be a random \mathbf{q} -Boltzmann planar map with perimeter 2ℓ distributed according to $\mathbb{P}^{(\ell)}$. The key is to use the fact that the area of $B^{(\ell)}$, appropriately rescaled, converges in distribution to \mathfrak{S} as $\ell \rightarrow \infty$. More precisely, it is proved in [16, Proposition 4] that we have the convergence in distribution

$$\frac{1}{\ell^{\theta+1/2}} \cdot |B^{(\ell)}| \xrightarrow[\ell \rightarrow \infty]{(d)} \mathbf{b}_\mathbf{q} \cdot \mathfrak{S} \quad \text{with} \quad \mathbf{b}_\mathbf{q} = \frac{2\gamma \cos(\pi(1 + \theta))}{c\sqrt{\pi}}. \quad (42)$$

Now imagine that we start a branching peeling exploration of $B^{(\ell)}$ which only peels along the locally largest hole until it stops (that is, the holes which are not the locally largest one are frozen and never explored afterwards, see the proof of Lemma 13 in [6] for a precise definition). If we denote by $V^{(\ell)}$ the number of internal vertices revealed during this exploration and if $(\delta_i^{(\ell)})_{i \geq 1}$ are the half-lengths of the holes which are frozen, ranked in decreasing order, then by Proposition 6.4 and Proposition 6.5 we have

$$|B^{(\ell)}| = V^{(\ell)} + \sum_{i \geq 1} |B_i^{(\delta_i^{(\ell)})}|,$$

where for every $i \geq 1$, the variable $|B_i^{(\delta_i^{(\ell)})}|$ has the law of the area of a \mathbf{q} -Boltzmann map of perimeter $2\delta_i^{(\ell)}$ and are independent conditionally on the exploration so far. Denote by $(\Delta_1, \Delta_2, \dots)$ the absolute values of the negative jumps of $t \mapsto X_\theta^{(-\theta)}(\mathbf{c}_q t)$ ranked in decreasing order. Using the fact that by Proposition 6.7, $\ell^{-1}(\delta_1^{(\ell)}, \delta_2^{(\ell)}, \dots)$ converges in distribution to $(\Delta_1, \Delta_2, \dots)$, using (42) it follows that we have the following inequality for the stochastic order

$$\mathfrak{S} \geq_{\text{sto}} \sum_{i \geq 0} \Delta_i^{\theta+1/2} \mathfrak{S}_i, \quad (43)$$

where $(\mathfrak{S}_i : i \geq 1)$ are i.i.d. copies of \mathfrak{S} . By (14) we also have $\sum_{i \geq 0} \mathbb{E}(\Delta_i^{\theta+1/2}) = 1$ (recall that in our setting $\omega_- = \theta + 1/2$), it follows that both sides of (43) have the same expectation, so that (43) is actually an equality in distribution. Hence \mathfrak{S} is a fixed point of the recursive distributive equation (14), which, as we have already seen, has a unique solution with given mean. Since \mathfrak{S} has mean 1, the conclusion follows. The case $\theta = \frac{3}{2}$ is established in a similar way, by using [21, Proposition 9] and [6, Proposition 9], instead of respectively [16, Proposition 4] and Proposition 6.7. \square

6.5 Slicing at heights large planar maps with high degrees

In this section, we give a more geometric flavor to our connection between growth-fragmentation processes and planar maps. We show that the scaling limit of the perimeters of the cycles obtained by slicing $B^{(\ell)}$ at fixed heights is a time-changed version of the growth-fragmentation appearing in the previous section. Unlike Proposition 6.7 which is valid for every $\theta \in (1/2, 1) \cup (1, 3/2)$, the results in this section only hold for the so-called dilute phase, where $\theta \in (1, 3/2)$. Before stating the main result, we introduce some notation.

If \mathbf{m} is a (bipartite) planar map, recall that \mathbf{m}^\dagger stands for the dual map of \mathbf{m} . If f is a face of \mathbf{m} , its height is by definition the dual graph distance $d_{\text{gr}}^\dagger(f, f_r)$ in \mathbf{m}^\dagger between f and the root face f_r . For $r \geq 0$, we let

$$\text{Ball}_r^\dagger(\mathbf{m})$$

be the map made by keeping inside \mathbf{m} all the faces at height less than or equal to r and cutting along all the edges which are adjacent on both sides to faces at height r . Equivalently, the corresponding connected subset $(\text{Ball}_r^\dagger(\mathbf{m}))^\circ$ of dual edges in \mathbf{m}^\dagger is given by those edges of \mathbf{m}^\dagger which contain at least one endpoint at height strictly less than r . Then $\text{Ball}_r^\dagger(\mathbf{m})$ is a submap of \mathbf{m} with possibly several holes. We denote by

$$\mathbf{L}(r) := (L_1(r), L_2(r), \dots),$$

the half-lengths of the cycles of $\text{Ball}_r^\dagger(\mathbf{m})$ ranked in decreasing order. An example is shown in Fig. 5.

Let $\mathbf{X}_\theta^{(1-\theta)}$ be the growth-fragmentation process with characteristics $(\kappa_\theta, 1 - \theta)$: it can be constructed from the process $X_\theta^{(1-\theta)}$ with characteristics $(\Psi_\theta, 1 - \theta)$ for the evolution of the Eve cell. Note that the

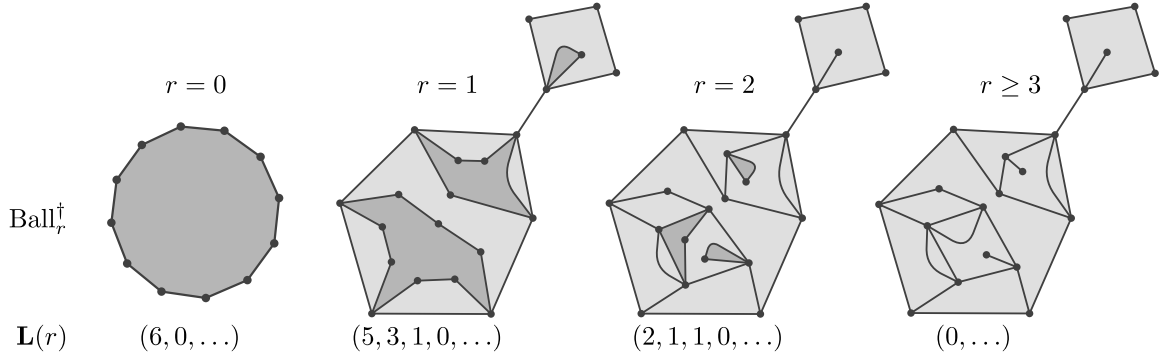


Figure 5: Example of the slicing at heights for a finite planar map (shown on the far right) with root face taken to be the outer face. The dark shaded faces correspond to the holes.

self-similar Markov processes $X_\theta^{(1-\theta)}$ and $X_\theta^{(-\theta)}$ (which was introduced just before Proposition 6.7) are related by the following time-change relation:

$$X_\theta^{(1-\theta)}(t) = X_\theta^{(-\theta)}\left(\int_0^t \frac{ds}{X_\theta^{(1-\theta)}(s)}\right), \quad t \geq 0.$$

Recall also from Sec. 4 and Sec. 4.3 the notation \mathbb{P}_1^+ and \mathbb{P}_1^- for the laws of the biased versions of the growth-fragmentation, and that $\mathbf{b}_\mathbf{q} = \frac{2\gamma \cos(\pi(1+\theta))}{c\sqrt{\pi}}$, $\mathbf{c}_\mathbf{q} = \frac{\pi c}{\Gamma(1+\theta) \cos((1+\theta)\pi)}$. Finally, set

$$\mathbf{a}_\mathbf{q} = \frac{1}{2} \left(1 + \sum_{k=0}^{\infty} (2k+1) \nu(k) \right)$$

and

$$\ell_{\theta+3/2}^\downarrow := \left\{ \mathbf{x} = (x_i)_{i \in \mathbb{N}} : x_1 \geq x_2 \geq \dots \geq 0 \text{ and } \sum_{i=1}^{\infty} x_i^{\theta+3/2} < \infty \right\}.$$

Theorem 6.10 (Slicing at heights). *Let $\mathbf{q} = (q_k)_{k \geq 1}$ be an admissible critical and non-generic weight sequence satisfying $q_k \sim c\gamma^{k-1}k^{-1-\theta}$ as $k \rightarrow \infty$, for $\theta \in (1, 3/2)$. Then the following three convergences hold in distribution*

$$\begin{aligned} \text{under } \mathbb{P}_\infty^{(\ell)}, \quad & \left(\frac{1}{\ell} \cdot \mathbf{L}(\ell^{\theta-1} \cdot t) : t \geq 0 \right) \xrightarrow[\ell \rightarrow \infty]{(d)} \left(\mathbf{X}_\theta^{(1-\theta)} \left(\frac{\mathbf{c}_\mathbf{q}}{\mathbf{a}_\mathbf{q}} \cdot t \right) : t \geq 0 \right) \text{ under } \mathbb{P}_1^+ \\ \text{under } \mathbb{P}^{(\ell)}, \quad & \left(\frac{1}{\ell} \cdot \mathbf{L}(\ell^{\theta-1} \cdot t) : t \geq 0 \right) \xrightarrow[\ell \rightarrow \infty]{(d)} \left(\mathbf{X}_\theta^{(1-\theta)} \left(\frac{\mathbf{c}_\mathbf{q}}{\mathbf{a}_\mathbf{q}} \cdot t \right) : t \geq 0 \right) \text{ under } \mathbb{P}_1 \\ \text{under } \mathbb{P}_\bullet^{(\ell)}, \quad & \left(\frac{1}{\ell} \cdot \mathbf{L}(\ell^{\theta-1} \cdot t) : t \geq 0 \right) \xrightarrow[\ell \rightarrow \infty]{(d)} \left(\mathbf{X}_\theta^{(1-\theta)} \left(\frac{\mathbf{c}_\mathbf{q}}{\mathbf{a}_\mathbf{q}} \cdot t \right) : t \geq 0 \right) \text{ under } \mathbb{P}_1^- \end{aligned} \quad (44)$$

in the space of càdlàg process taking values in $\ell_{\theta+3/2}^\downarrow$ equipped with the Skorokhod J_1 topology. In addition, if $\mathcal{M}_\theta^-(\infty)$ is the intrinsic area of $\mathbf{X}_\theta^{(1-\theta)}$ under \mathbb{P}_1 , then the convergence

$$\frac{1}{\ell^{\theta+1/2}} \cdot |\mathbf{m}| \xrightarrow[\ell \rightarrow \infty]{} \mathbf{b}_\mathbf{q} \cdot \mathcal{M}_\theta^-(\infty) \quad (45)$$

holds jointly in distribution with (44).

Let us first explain why we need to restrict to the case $\theta \in (1, 3/2)$ for this theorem. Let $B^{(\ell),\dagger}$ be the dual of a random map distributed according to $\mathbb{P}^{(\ell)}$. The second convergence above as well as the results of [16] suggest that the typical distances in $B^{(\ell),\dagger}$ are of order $\ell^{\theta-1}$ when $\theta \in (1, 3/2)$. However, in the dense case $\theta \in (1/2, 1)$ the results of [16] suggest that this typical distance should be of order $\log(\ell)$. This logarithmic scaling prevents us from expecting a non-trivial self-similar object in the limit and this is why we restrict our attention to the dilute case $\theta \in (1, 3/2)$.

The proof of this theorem follows the same lines as that of [6] and we will only sketch its proof. We first introduce the peeling algorithm that we use to study the geometric structure of the dual of \mathbf{q} -Boltzmann planar maps, and which is adapted from [16].

Let \mathbf{m} be a rooted bipartite planar map. Recall that a peeling exploration starts with \mathbf{e}_0 , the map consisting of the root face of \mathbf{m} (seen as a simple cycle). Inductively suppose that at step $i \geq 0$, the following hypothesis is satisfied:

(H): There exists an integer $h \geq 0$ such all the faces adjacent to the holes of \mathbf{e}_i are at height h or $h+1$ in \mathbf{m} . Suppose furthermore that the faces adjacent to a same hole in \mathbf{e}_i and which are at height h form a connected part on the boundary of that hole.

We will take the height of an edge of the boundary of a hole of \mathbf{e}_i to mean the height of its incident face. If (H) is satisfied by \mathbf{e}_i the next edge to peel $\mathcal{L}^\dagger(\mathbf{e}_i)$ is chosen as follows:

- If all edges on the boundaries of the holes of \mathbf{e}_i are at height h then $\mathcal{L}^\dagger(\mathbf{e}_i)$ is a deterministic edge on the boundary of one of its hole,
- Otherwise $\mathcal{L}^\dagger(\mathbf{e}_i)$ is a deterministic edge at height h such that the next edge in clockwise order around its hole is at height $h+1$.

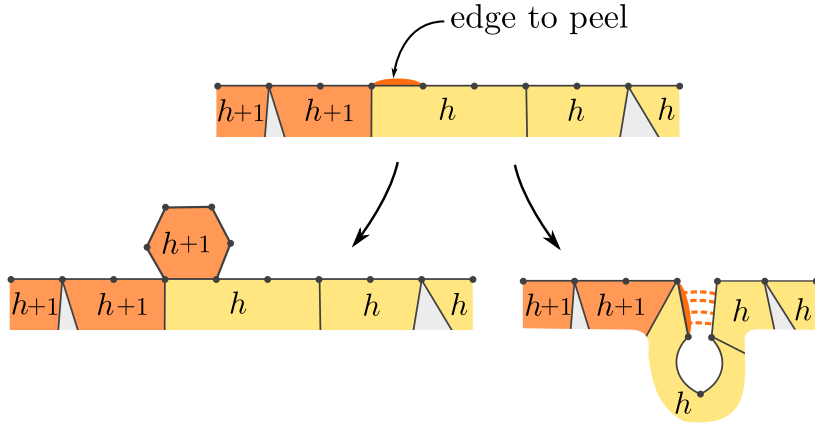


Figure 6: Illustration of the peeling using algorithm \mathcal{L}^\dagger .

It is easy to check by induction that if one uses the above algorithm starting at step $i = 0$ to peel the edges of \mathbf{m} , then for every $i \geq 0$ the explored map \mathbf{e}_i satisfies the hypothesis (H) and one can indeed define the peeling exploration of \mathbf{m} using the algorithm \mathcal{L}^\dagger . Let us give a geometric interpretation of this peeling exploration. We denote by $H(\mathbf{e}_i)$ the minimal height in \mathbf{m} of a face adjacent to one of its holes and let $\theta_r = \inf\{i \geq 0 : H(\mathbf{e}_i) = r\}$ for $r \geq 0$. We easily prove by induction on $r \geq 0$ that:

$$\mathbf{e}_{\theta_r} = \text{Ball}_r^\dagger(\mathbf{m}). \quad (46)$$

Sketch of the proof of Theorem 6.10. Let us focus first on the second convergence which is the extension of the convergence proved in [6]. We sample a random planar map distributed according to $\mathbb{P}^{(\ell)}$ and explore its dual metric structure using the above peeling exploration. As in [6, Sec. 2.6], during this exploration we first track the evolution of the locally largest cycle \mathcal{C}_* starting with the boundary of the map and we freeze all the other cycles when created. By Proposition 6.4 we already know that the scaling limit for the half-perimeter P_* of this cycle as a function of the number of peeling steps is given by the process $X_\theta^{(-\theta)}$. Our first goal is to prove that the scaling limit of P_* in the *height parameter* is given by the process $X_\theta^{(1-\theta)}$. This is proved as in [6, Proposition 12 and Lemma 13] using the work [16] instead of [21] as the key input. Specifically, if $\tilde{P}_*(h)$ is the half-perimeter of the locally largest cycle (that starts with the boundary of the map) when height $h \geq 0$ is reached then under $\mathbb{P}^{(\ell)}$ we have

$$\left(\frac{1}{\ell} \tilde{P}_*(\lfloor \ell^{\theta-1} \cdot t \rfloor) \right)_{t \geq 0} \xrightarrow[\ell \rightarrow \infty]{(d)} \left(X_\theta^{(1-\theta)}(c_q t / a_q) \right)_{t \geq 0}, \quad (47)$$

in distribution for the Skorokhod J_1 topology. Once this is granted we perform an exploration with a cutoff $\varepsilon > 0$ exactly as in [6, Sec. 3.3] which consists roughly speaking in exploring the map with the above peeling algorithm but freezing those cycles whose half-perimeters drop below $\varepsilon \ell$. We then use the Markov property of the underlying random map and (47) to deduce that after performing this cutoff we have convergence of the non-frozen cycles at heights towards the cutoff version of the growth-fragmentation process $\mathbf{X}_\theta^{(1-\theta)}$, see [6, Corollary 18]. The point is that to perform these explorations with cutoff we only need to follow a tight number of locally largest cycles. Finally, we need to control, in the $\ell_{\theta+3/2}$ -sense, the error made by the cutoff procedure both on the planar map side (see [6, Proposition 15]) and on the growth-fragmentation side (see [6, Lemma 22]). Here again, the proofs are easily adapted from [6] to our case using the discrete and continuous martingales developed in this text rather than those used in [6].

For the first and third convergence we proceed similarly except that instead of first following the locally largest cycle that starts with the boundary of the map, we first follow the distinguished cycles \mathcal{C}_∞ in the first case and \mathcal{C}_\bullet in the third case. We can then establish the analogs of (47) involving respectively the processes Y^+ and Y^- associated with $\mathbf{X}_\theta^{(1-\theta)}$ using similar arguments as above together with Proposition 6.7. The rest of the proof goes through: we fix $t_0 > 0$ and perform a cut-off exploration to prove that we have convergence of the non-frozen cycles at heights smaller than $\ell^{\theta-1} t_0$ towards the cutoff version of the growth-fragmentation process $(\mathbf{X}_\theta^{(1-\theta)}(s) : 0 \leq s \leq t_0)$ under \mathbb{P}_1^+ and \mathbb{P}_1^- respectively. The $\ell^{\theta+3/2}$ -control of the error made is then transferred from the previous case using absolute continuity relations between the laws of $(\mathbf{L}(h) : 0 \leq h \leq \ell^{\theta-1} \cdot t_0)$ under $\mathbb{P}^{(\ell)}, \mathbb{P}_\bullet^{(\ell)}$ and $\mathbb{P}_\infty^{(\ell)}$ and similarly between the laws of $(\mathbf{X}_\theta^{(1-\theta)}(s) : 0 \leq s \leq t_0)$ under $\mathbb{P}_1, \mathbb{P}_1^+$ and \mathbb{P}_1^- .

Let us now focus on the second part of the theorem. The main idea to show that the convergences (44) and (45) hold jointly in distribution is to approximate both the number of vertices of a large Boltzmann planar map and the intrinsic area of the growth-fragmentation by respectively a finite number of cycles and a finite number of cells.

To this end, we first introduce some preliminary notation. For fixed $\varepsilon > 0$, under $\mathbb{P}^{(\ell)}$, we perform a peeling exploration of a map \mathbf{m} but we freeze all the cycles of half-perimeter less than $\varepsilon \ell > 0$ when they appear (by convention, we say that an inner vertex is a cycle of length 0 and that under $\mathbb{P}^{(0)}$ the map has size one almost surely). We denote by $\ell_1^{(\varepsilon \ell)}, \dots, \ell_k^{(\varepsilon \ell)}, \dots$ the half-perimeters of the frozen cycles ranked in decreasing order. Similarly, in the continuous counterpart, under \mathcal{P}_1 , we freeze the evolution of a cell of the cell system as soon as its size becomes less than ε , and we denote by $(L_i^{(\varepsilon)})_{i \geq 1}$ the sizes of these frozen cells ranked in decreasing order.

We will rely on the following lemma, which, roughly speaking, says that very small cycles do not

contribute much to the total area.

Lemma 6.11. *For every $\delta > 0$, we can find $\varepsilon > 0$ such that*

$$\limsup_{\ell \rightarrow \infty} \mathbb{E}^{(\ell)} \left(\frac{1}{f^\vee(\ell)} \sum_{i \geq 1} f^\vee(\ell_i^{(\varepsilon \ell)}) \mathbb{1}_{\ell_i^{(\varepsilon \ell)} \leq \varepsilon^2 \ell} \right) \leq \delta \quad (48)$$

and

$$\mathcal{E}_1 \left(\sum_{i \geq 1} (L_i^{(\varepsilon)})^{\omega_-} \mathbb{1}_{L_i^{(\varepsilon)} \leq \varepsilon^2} \right) \leq \delta. \quad (49)$$

Before proving these estimates, we explain how to finish the proof of Theorem 6.10 and prove that the convergences (44) and (45) hold jointly in distribution. First, in the notation of Proposition 2.5, note that

$$\mathbf{M}_\varepsilon^- = \sum_{i \geq 1} (L_i^{(\varepsilon)})^{\omega_-}, \quad (50)$$

and that the proof of Proposition 2.5 shows that we can write

$$\mathcal{M}_\theta^-(\infty) = \sum_{i \geq 1} (L_i^{(\varepsilon)})^{\omega_-} \cdot \xi_i, \quad (51)$$

where the random variables $(\xi_i)_{i \geq 1}$ are i.i.d. random variables distributed as $\mathcal{M}_\theta^-(\infty)$ and independent of $(L_i^{(\varepsilon)} : i \geq 1)$. In particular, $\mathbb{E}(\xi_i) = 1$.

Now denote by $\ell_1^{(\varepsilon \ell, \varepsilon^2 \ell)}, \dots, \ell_k^{(\varepsilon \ell, \varepsilon^2 \ell)}, \dots$ the sub-collection of the frozen cycles of half-perimeter at least $\varepsilon^2 \ell$ ranked in decreasing order. We also let $(L_i^{(\varepsilon, \varepsilon^2)})_{i \geq 1}$ be the sub-collection of the elements of $(L_i^{(\varepsilon)})_{i \geq 1}$ which are at least ε^2 , ranked in decreasing order. Since the collection $(L_i^{(\varepsilon, \varepsilon^2)})_{i \geq 1}$ is almost surely finite (if it were infinite, the quantity \mathbf{M}_ε^- would be infinite), by Proposition 6.7 we have the following convergence in distribution:

$$\frac{1}{\ell} \cdot (\ell_i^{(\varepsilon \ell, \varepsilon^2 \ell)})_{i \geq 1} \xrightarrow[\ell \rightarrow \infty]{(d)} (L_i^{(\varepsilon, \varepsilon^2)})_{i \geq 1}. \quad (52)$$

To establish the desired joint convergence, it is therefore enough to check that for every $\delta > 0$ there exists $\varepsilon > 0$ such that

$$\limsup_{\ell \rightarrow \infty} \mathbb{P}^{(\ell)} \left(\frac{1}{f^\vee(\ell)} \left| |\mathbf{m}| - \sum_{i \geq 1} f^\vee(\ell_i^{(\varepsilon \ell, \varepsilon^2 \ell)}) \right| \geq \delta \right) \leq \delta \quad (53)$$

and

$$\mathcal{P}_1 \left(\left| \mathcal{M}_\theta^-(\infty) - \sum_{i \geq 1} (L_i^{(\varepsilon, \varepsilon^2)})^{\omega_-} \right| \geq \delta \right) \leq \delta. \quad (54)$$

For (54), by using the identity (50), write

$$\mathcal{P}_1 \left(\left| \mathcal{M}_\theta^-(\infty) - \sum_{i \geq 1} (L_i^{(\varepsilon, \varepsilon^2)})^{\omega_-} \right| \geq \delta \right) \leq \mathcal{P}_1(|\mathcal{M}_\theta^-(\infty) - \mathbf{M}_\varepsilon^-| \geq \delta/2) + \mathcal{P}_1 \left(\sum_{i \geq 1} (L_i^{(\varepsilon)})^{\omega_-} \mathbb{1}_{L_i^{(\varepsilon)} \leq \varepsilon^2} \geq \delta/2 \right).$$

By choosing $\varepsilon > 0$ small enough, the first term of the right-hand side of the last inequality may be made arbitrarily small by Proposition 2.5, the second one by combining the Markov inequality with (49).

We now focus on (53). By injecting (51) in (54), we get the existence of $\varepsilon > 0$ such that

$$\mathcal{P}_1 \left(\left| \sum_{i \geq 1} (L_i^{(\varepsilon)})^{\omega_-} \cdot \xi_i - \sum_{i \geq 1} (L_i^{(\varepsilon, \varepsilon^2)})^{\omega_-} \right| \geq \delta \right) \leq \delta.$$

As before, the Markov inequality combined with an appeal to (49) (and using the fact that $\mathbb{E}(\xi_i) = 1$) shows that up to diminishing our $\varepsilon > 0$ we have

$$\mathcal{P}_1 \left(\left| \sum_{i \geq 1} (L_i^{(\varepsilon, \varepsilon^2)})^{\omega_-} \cdot \xi_i - \sum_{i \geq 1} (L_i^{(\varepsilon, \varepsilon^2)})^{\omega_-} \right| \geq \delta \right) \leq \delta.$$

Similarly, in the discrete setting, under $\mathbb{P}^{(\ell)}$ and conditionally given $(\ell_i^{(\varepsilon \ell)} : i \geq 1)$, $|\mathbf{m}|$ has the same distribution as $\sum_{i \geq 1} \mathcal{V}_i(\ell_i^{(\varepsilon \ell)})$, where $(\mathcal{V}_i(m) : i \geq 1, m \geq 1)$ are independent variables distributed as the number of vertices of a Boltzmann map with perimeter $2m$ and which are independent of $(\ell_i^{(\varepsilon \ell)} : i \geq 1)$. Using the last display, (52) and the fact that $(f^\vee(m))^{-1} \cdot \mathcal{V}_i(m)$ converges in distribution to ξ_i as $m \rightarrow \infty$ (see [16, Proposition 4]), we deduce that there exists $\varepsilon > 0$ such that

$$\limsup_{\ell \rightarrow \infty} \mathbb{P}^{(\ell)} \left(\left| \frac{1}{f^\vee(\ell)} \sum_{i \geq 1} \mathcal{V}_i(\ell_i^{(\varepsilon \ell, \varepsilon^2 \ell)}) - \sum_{i \geq 1} f^\vee(\ell_i^{(\varepsilon, \varepsilon^2)}) \right| \geq \delta \right) \leq \delta. \quad (55)$$

Since $\mathbb{E}(\mathcal{V}_i(k)) = f^\vee(k)$ (by (41)), we have

$$\mathbb{E}^{(\ell)} \left(\frac{1}{f^\vee(\ell)} \sum_{i \geq 1} \mathcal{V}_i(\ell_i^{(\varepsilon \ell)}) \mathbb{1}_{\ell_i^{(\varepsilon \ell)} \leq \varepsilon^2 \ell} \right) = \mathbb{E}^{(\ell)} \left(\frac{1}{f^\vee(\ell)} \sum_{i \geq 1} f^\vee(\ell_i^{(\varepsilon \ell)}) \mathbb{1}_{\ell_i^{(\varepsilon \ell)} \leq \varepsilon^2 \ell} \right).$$

By combining this with (55), we can use (48) twice to transform in (55) the appearances of $\ell_i^{(\varepsilon \ell, \varepsilon^2 \ell)}$ into $\ell_i^{(\varepsilon \ell)}$ to finally get (53). This completes the proof. \square

It remains to establish Lemma 6.11 and we start with yet another lemma. Recall that Υ_θ^\vee is a θ -stable Lévy process with positivity parameter ρ such that $\theta(1 - \rho) = 1/2$, conditioned to die continuously at 0 when it enters \mathbb{R}_- .

Lemma 6.12. *For every $y > 0$, set $\sigma_y^- = \inf\{t : \Upsilon_\theta^\vee(t) \leq y\}$. Then*

$$\mathbb{P} \left(\Upsilon_\theta^\vee(\sigma_\varepsilon^-) \leq \varepsilon^2 \right) \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Proof. Let ξ^\vee be the Lévy process appearing in the Lamperti representation of Υ_θ^\vee . Setting $T_y^- = \inf\{t : \xi^\vee \leq y\}$ for $y \in \mathbb{R}$, it follows that, for every $\varepsilon > 0$,

$$\mathbb{P} \left(\Upsilon_\theta^\vee(\sigma_\varepsilon^-) \leq \varepsilon^2 \right) = \mathbb{P} \left(\xi^\vee(T_{\ln(\varepsilon)}^-) \leq 2 \ln(\varepsilon) \right).$$

By the second identity of Corollary 5 in [20] (where one should read ξ^\vee instead of ξ^\uparrow), we have (recall that our positivity parameter ρ satisfies $\theta(1 - \rho) = 1/2$, and that in [20], ρ stands for $1 - \rho$) for every $v < 0$:

$$\mathbb{P} \left(v - \xi^\vee(T_v^-) \in dx \right) = \frac{1}{\pi} \sqrt{\frac{1 - e^v}{e^x - 1}} \cdot \frac{1}{1 - e^{v-x}} dx.$$

Therefore, for every $\varepsilon \in (0, 1)$,

$$\mathbb{P} \left(\xi^\vee(T_{\ln(\varepsilon)}^-) \leq 2 \ln(\varepsilon) \right) = \frac{1}{\pi} \int_{\ln(1/\varepsilon)}^\infty \sqrt{\frac{1 - \varepsilon}{e^x - 1}} \cdot \frac{1}{1 - \varepsilon \cdot e^{-x}} dx = 1 - \frac{2}{\pi} \text{ArcCot}(\sqrt{\varepsilon}),$$

where ArcCot denotes the inverse of $\cot(z) = 1/\tan(z)$. This tends to 0 as $\varepsilon \rightarrow 0$, and the proof is complete. \square

Proof of Lemma 6.11. We start by showing (49). An extension of Proposition 4.6 to stopping times yields the identity

$$\mathcal{E}_1 \left(\sum_{i \geq 1} (L_i^{(\varepsilon)})^{\omega_-} \mathbb{1}_{L_i^{(\varepsilon)} \leq \varepsilon^2} \right) = \mathbb{P} \left(\Upsilon_\theta^\vee(\sigma_\varepsilon^-) \leq \varepsilon^2 \right).$$

The estimate (49) then follows from Lemma 6.12.

The proof of (48) mimics this argument in the discrete setting. Since there are no inner vertices in the map explored with the peeling frozen at level $\varepsilon\ell$ (recall that an inner vertex is seen as a cycle of length 0), we can write

$$\frac{1}{f^\vee(\ell)} \mathbb{E}^{(\ell)} \left(\sum_{i \geq 1} f^\vee(\ell_i^{(\varepsilon\ell)}) \mathbb{1}_{\{\ell_i^{(\varepsilon\ell)} \leq \varepsilon^2 \ell\}} \right) = \frac{1}{f^\vee(\ell)} \mathbb{E}^{(\ell)} \left(\sum_{x \in \text{Vertices}(\mathfrak{m})} \mathbb{1}_{\{\mathcal{L}(x, \varepsilon) \leq \varepsilon^2 \ell\}} \right),$$

where $\mathcal{L}(x, \varepsilon)$ is the first cycle surrounding x which, during the exploration of the map, drops below a half-perimeter of $\varepsilon\ell$. The extension of Proposition 6.4 to stopping times yields the identity

$$\frac{1}{f^\vee(\ell)} \mathbb{E}^{(\ell)} \left(\sum_{x \in \text{Vertices}(\mathfrak{m})} \mathbb{1}_{\{\mathcal{L}(x, \varepsilon) \leq \varepsilon^2 \ell\}} \right) = \mathbb{P}_\bullet^{(\ell)}(P_\bullet(\tau_{\varepsilon\ell}) \leq \varepsilon^2 \ell)$$

where we recall that P_\bullet is the evolution of the half-perimeter of the special cycle surrounding the distinguished point of the map and where we put $\tau_{\varepsilon\ell}$ for the first time this half-perimeter drops below $\varepsilon\ell$. By Proposition 6.7,

$$\mathbb{P}_\bullet^{(\ell)}(P_\bullet(\tau_{\varepsilon\ell}) \leq \varepsilon^2 \ell) \xrightarrow{\ell \rightarrow \infty} \mathbb{P} \left(\Upsilon_\theta^\vee(\sigma_\varepsilon^-) \leq \varepsilon^2 \right).$$

The estimate (48) then follows from Lemma 6.12, and this completes the proof. \square

For $\theta \in (1, 3/2)$, we believe that the growth-fragmentation $\mathbf{X}_\theta^{(1-\theta)}$ should describe so-called stable disks, which are the conjectural scaling limits of the maps $\ell^{-\theta+1} \cdot B^{(\ell), \dagger}$. Similarly to the fact that the Brownian disk may be seen as a Brownian map with a boundary, we believe that stable disks may be seen as stable spheres with a boundary, which should be new compact metric spaces, described by the growth-fragmentation $\mathbf{X}_\theta^{(1-\theta)}$ started from 0. In particular, this would lead to examples of continuum random surface models with other scaling exponents than those of the Brownian map. Indeed, roughly speaking, when scaling distances by a factor C in the Brownian map, the area measure is multiplied by C^4 while lengths of outer boundaries of metric balls are multiplied by C^2 . In the case of a stable sphere with $1 < \theta < 3/2$, we expect that C^4 will be replaced with $C^{(\theta+1/2)/(\theta-1)}$ and C^2 with $C^{1/(\theta-1)}$.

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